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A NEW LOSS FUNCTİON BASED ON BURR XII FOR USİNG RİSK ESTİMATİON

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Achieving a competitive advantage in today's fast-paced and globally competitive business environment is one of the main goals for organizations across all industries. To attain this, businesses must adopt effective strategies and tools to enhance their operational efficiency, product quality, and overall customer satisfaction. One of the most significant tools that can help businesses achieve this goal is the implementation of quality control methods. Quality control not only ensures that products and services meet predefined standards but also helps reduce costs, increase customer loyalty, and maintain a sustainable competitive edge. In recent years, the importance of loss functions in quality assurance has grown considerably. Loss functions are used to quantify the deviation of a product's performance or quality from its desired target, translating this deviation into a monetary loss. This concept enables businesses to assess the broader impact of poor quality, not only on the organization but also on society as a whole. The monetary value of the loss represents the cost associated with a product's failure to meet expectations, including customer dissatisfaction, warranty claims, and potential reputational damage. Advancements in statistical methodologies, particularly those involving inverted probability density functions (PDFs), have opened new avenues for the application of loss functions. Inverted PDFs allow for a more detailed understanding of quality-related losses. This paper introduces the Inverted Burr XII Loss Function (IBXIILF) as a novel member of the inverted probability loss function family. The IBXIILF provides a robust framework for evaluating and minimizing quality-related losses in various industrial settings. The performance and applicability of the IBXIILF are demonstrated through a comparative study and an industrial example, highlighting its practical relevance and effectiveness in monitoring losses.

Keywords: quality loss functions, inverted Burr XII loss function, risk function, inverted probability loss functions, Burr XII distribution

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1. INTRODUCTION

Loss functions play a critical role in decision theory and quality assurance by quantifying the losses associated with deviations from a desired target value. Introduced by Taguchi (1986), the concept of quality loss functions revolutionized quality control, with the quadratic loss function becoming widely recognized for its simplicity and practicality. However, the quadratic loss function faced criticism for its unbounded nature and potential to misrepresent economic losses under certain conditions. In response to these limitations, Spiring (1993) proposed the Inverted Normal Loss Function (INLF) based on the inversion of the normal probability density function (pdf). The INLF provided a bounded and more economically realistic assessment of losses due to target deviations. Sun (1996) further refined the INLF, enhancing its applicability and robustness. Subsequently, Drain and Gough (1996) introduced modified versions of these functions, expanding their versatility. Building on this foundation, Spiring and Yeung (1998) introduced a broader class of Inverted Probability Loss Functions (IPLFs) capable of handling both symmetric and asymmetric loss scenarios. Leung and Spiring (2004) further explored the properties of IPLFs, providing insights into their flexibility and effectiveness in diverse applications. More recently, Köksoy et al. (2019) introduced the Inverted Wald Loss Function (IWLF), extending the IPLF family with novel characteristics tailored to specific industrial contexts.

The Burr XII distribution, a member of the Burr family proposed by Irving W. Burr in 1942, offers remarkable adaptability for data fitting. This family of distributions, derived from solving differential equations, consists of twelve cumulative distribution functions, each tailored for various types of data modeling (Bismi, G.N., 2005). The Burr XII distribution's flexibility makes it a suitable candidate for constructing loss functions that can accommodate both symmetric and asymmetric processes.

This paper introduces the Inverted Burr XII Loss Function (IBXIILF) as a novel member of the IPLF family, leveraging the Burr XII probability density function (pdf). The IBXIILF addresses limitations in existing loss functions and offers versatility in adapting to both symmetric and asymmetric processes, broadening its applicability in quality assurance and decision-making contexts.

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The remainder of the paper is structured as follows: Section 2 provides a brief overview of the IPLF family. Section 3 presents the IBXIILF, including its derivation, properties, and 4 and 5 performance evaluation through illustrative examples and a comparative study. Finally, the paper concludes with a summary of findings and directions for future research.

2. BASIS OF INVERTED PROBABILITY LOSS FUNCTION

Loss functions are crucial tools in quantifying deviations from a desired target value as economic losses. Taguchi's quadratic loss function, while foundational, has faced criticism due to its unbounded nature, which implies infinitely increasing losses for deviations. This characteristic does not align well with real-world scenarios, where excessively large deviations often result in similar outcomes for consumers, regardless of their magnitude. To address this limitation, Spiring (1993) proposed the Inverted Normal Loss Function (INLF), a bounded loss function derived from the inversion of the normal probability density function (pdf). The INLF offers a finite maximum loss and a more realistic approach to economic loss assessment, making it preferable in many quality assurance contexts.

Building on this idea, Spiring and Yeung (1998) and Leung and Spiring (2004) introduced a family of Inverted Probability Loss Functions (IPLFs) based on the inverted structures of various probability distributions. These functions expanded the flexibility of loss modeling by incorporating both symmetric and asymmetric distributions. For example, IPLFs were developed using gamma and beta distributions to account for asymmetric loss scenarios in addition to the symmetric structure provided by the normal distribution. This versatility has made IPLFs an invaluable tool for applications requiring precise loss estimation.

The mathematical foundation of the IPLF family begins with defining $f(y)$ as a probability function with a single maximum point equal to the target value τ. The following equations define the components of the loss function:

$$\pi(y,\tau) = f(y) \tag{1}$$

$$m = sup_{y\in Y} f(y) = f(\tau) \tag{2}$$

The loss function is then defined as:

$$L(y,\tau) = K\left(1 - \frac{\pi(y,\tau)}{m}\right) \tag{3}$$

Here, K represents the maximum loss, and ($\frac{\pi(y,\tau)}{m}$), known as the Loss Inversion Ratio (LIR), varies between 0 and 1. The bounded nature of this structure ensures that loss is finite, addressing the shortcomings of unbounded loss functions.

2.1 Examples of IPLFs in the Literature

Inverted Normal Loss Function (INLF)

The INLF, first proposed by Spiring (1993), is symmetric and closely resembles the shape of the quadratic loss function but offers the advantage of bounded loss. Its functional form is:

$$L_{IN}(y) = K\left\{1 - \exp\left(-\frac{(y-\tau)^2}{2{\sigma_L}^2}\right)\right\} \tag{4}$$

where ${\sigma_L}^2$ is the scale parameter. The INLF's bounded nature makes it particularly effective for scenarios requiring more realistic loss modeling.

Inverted Gamma Loss Function (IGLF)

Spiring and Yeung (1998) introduced the IGLF, which accounts for asymmetric losses using the gamma distribution's pdf. Its form is:

$$L_{IG}(y,\tau) = K\left[1 - \left(\frac{y}{\tau}\exp\left(1 - \frac{y}{\tau}\right)\right)^{\alpha_L - 1}\right] \tag{5}$$

Here, α_L is the shape parameter for the loss function, providing flexibility in modeling varying levels of sensitivity to deviations.

Inverted Beta Loss Function (IBLF)

Leung and Spiring (2004) extended the IPLF family with the IBLF, derived from the beta distribution's pdf:

$$L_{IB}(y,\tau) = K\left(1-\left(\tau(1-\tau)^{\frac{(1-\tau)}{\tau}}\right)^{1-\alpha_L}\left(y(1-y)^{\frac{(1-\tau)}{\tau}}\right)^{\alpha_L-1}\right) \tag{6}$$

Here, α_L is the shape parameter for the loss function, providing flexibility in modeling varying levels of sensitivity to deviations.

Inverted Wald Loss Function (IWLF)

Recently, Köksoy et al. (2019) introduced the IWLF, an alternative to IGLF, specifically designed for right-skewed data using the Wald distribution. Its form is:

$$L_{IW}(y,\tau) \quad = K\left\{1-\frac{\sqrt{\tau^3}}{\sqrt{y^3}}exp\left(\left(\frac{-\ (\lambda_L-3\tau)}{2\tau^2}\right)(y-\tau)-\left(\frac{\lambda_L}{2}\right)\left(\frac{1}{y}-\frac{1}{\tau}\right)\right)\right\} \tag{7}$$

Here, λ_L is the scale parameter, allowing customization to reflect losses accurately in specific contexts.

Expected Loss and Risk Function

Loss functions not only quantify economic losses but also enable the computation of expected losses, representing average economic impact. The expected loss is calculated as:

$$EL(y) = \int_{-\infty}^{\infty} L(y)f(y)\,dy \tag{8}$$

where $L(y)$is the loss function, and $f(y)$ is the process's pdf. Accurate loss calculations depend on selecting the correct distribution for the process. Incorrect assumptions can lead to underestimation or overestimation of losses.

This study introduces the Inverted Burr XII Loss Function (IBXIILF), a novel addition to the IPLF family. Derived from the Burr XII pdf, the IBXIILF is designed to handle both symmetric and asymmetric processes, expanding the applicability of IPLFs in quality assurance and decision-making contexts.

3. THE PROPOSED NEW MEMBER OF THE FAMILY: INVERTED BURR XII LOSS FUNCTION

This section introduces the Inverted Burr XII Loss Function (IBXIILF), a novel loss function derived from the Burr XII probability density function (pdf). The IBXIILF is designed to adapt to both symmetric and asymmetric processes, providing a flexible alternative to existing loss functions in the literature. The Burr XII distribution, first proposed by Irving W. Burr in 1942, belongs to a family of cumulative distribution functions known for their versatility in modeling diverse datasets, especially when normality assumptions are invalid. The Burr XII distribution is defined as follows:

Cumulative Distribution Function (CDF):

$$F(y) = 1-\frac{1}{\left(1+\left(\frac{y}{\alpha}\right)^{c}\right)^{k}}, \qquad x \geq 0 \tag{9}$$

Probability Density Function (PDF):

$$f(y) = \frac{ck}{\alpha}\left(\frac{y}{\alpha}\right)^{c-1}\left(1+\left(\frac{y}{\alpha}\right)^{c}\right)^{-(k+1)} \tag{10}$$

Here, c and k are the shape parameters, and α is the scale parameter of the distribution. Over time, the Burr XII distribution has been recognized as a special case or limiting distribution of widely used theoretical distributions such as Normal, Exponential, Weibull, and Extreme Value distributions (Tadikamalla, 1980; Rodriguez, 1977 and Lewis, 1981).

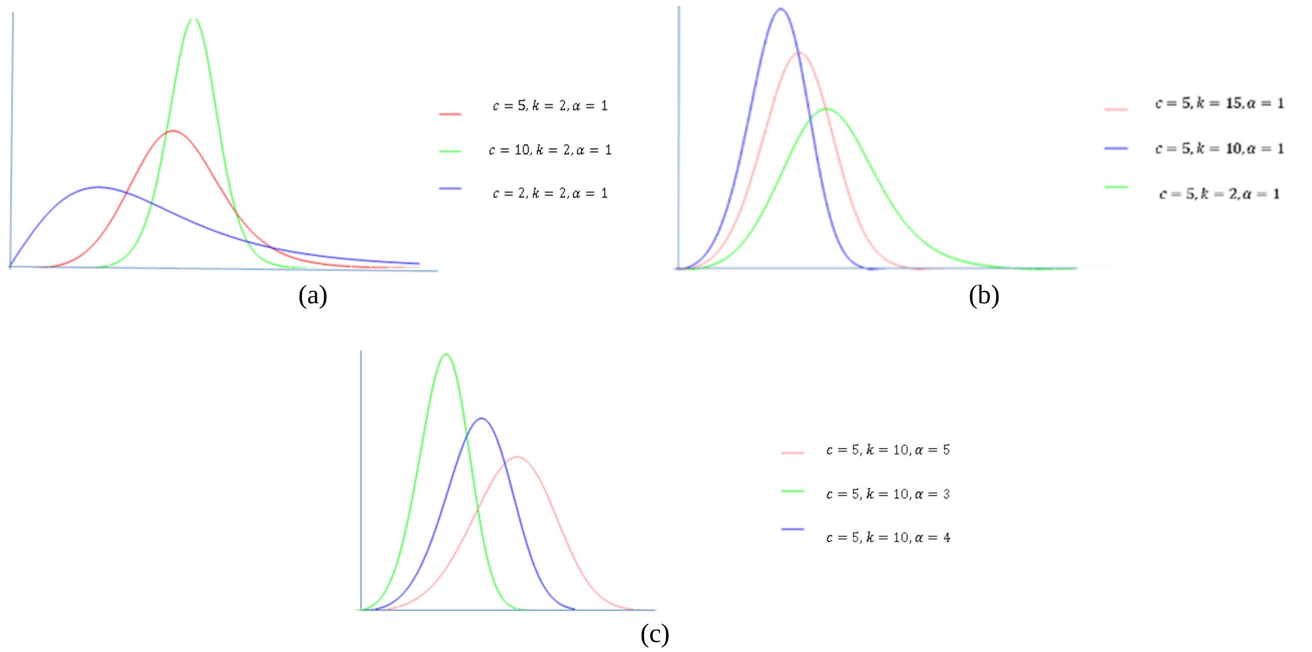

Figure 1. Illustration of Burr XII distribution for different values of the c, k and α parameter

The Burr XII distribution, as illustrated in Figure 1(a), Figure 1(b), and Figure 1(c), demonstrates remarkable flexibility in matching the shapes of many known distribution families under varying parameters. This versatility makes the Burr XII distribution a valuable tool for developing new theories and methodologies, particularly in quality research and development. Its ability to address scenarios where normality assumptions fail—such as with distorted data—can significantly improve the accuracy of measurement results.

In the realm of loss functions, Spiring (1993) proposed the Inverted Normal Loss Function (INLF), a finite loss function derived by inverting the normal distribution pdf, as an alternative to the quadratic loss function. While the quadratic loss function assumes infinite loss, INLF provides a more realistic assessment. Sun et al. (1996) further refined INLF by introducing a more flexible variant, MINLF. Later, Leung and Spiring (2004) expanded the family of loss functions by introducing IBLF and IGLF, which are based on the probability density functions of beta and gamma distributions, respectively, accommodating asymmetric structures. Köksoy et al. (2019) introduced IWLF, an alternative to IGLF, specifically designed for right-skewed data based on the Wald distribution pdf.

Building on these advancements, this paper introduces a novel loss function derived from the Burr XII pdf. This new function is unique in its adaptability to both symmetric and asymmetric processes, positioning it as a comprehensive alternative to existing loss functions. The Burr XII distribution (α,c,k) enhances the precision of loss measurement due to its exceptional ability to fit diverse data distributions effectively.

Derivation of the Inverted Burr XII Loss Function (IBXIILF)

The Inverted Burr XII Loss Function (IBXIILF) is derived from the Burr XII probability density function (pdf). The mode of the Burr XII distribution, where the pdf reaches its maximum value, is defined as the target value τ

$$\tau = \alpha\left(\frac{c-1}{ck+1}\right)^{1/c} \tag{11}$$

By rearranging Equation (11), the parameter k can be expressed as: $k = \frac{\alpha^c.c-\alpha^c-\tau^c}{\tau^c.c}$.
Using Equation (1), the probability density ratio $\pi(y,\tau)$ is obtained as:

$$\pi(y,\tau) \;=\; \frac{\left(\frac{\alpha^c.c-\alpha^c-\tau^c}{\tau^c.c}\right)}{\alpha} c\left(\frac{y}{\alpha}\right)^{c-1} \Bigg/ \left(1+\left(\frac{y}{\alpha}\right)^c\right)^{\frac{\alpha^c.c-\alpha^c-\tau^c}{\tau^c.c}+1} \tag{12}$$

$$\mathrm{m} = \; sup_{y\in Y} f(y) = f(\tau) = \frac{\left(\frac{\alpha^c.c-\alpha^c-\tau^c}{\tau^c.c}\right)}{\alpha} c\left(\frac{\tau}{\alpha}\right)^{c-1} \Bigg/ \left(1+\left(\frac{\tau}{\alpha}\right)^c\right)^{\frac{\alpha^c.c-\alpha^c-\tau^c}{\tau^c.c}+1} \tag{13}$$

The Loss Inversion Ratio (LIR) is calculated using Equations (12) and (13):

$$\frac{\pi(y,\tau)}{m} = \left(\frac{y}{\tau}\right)^{c-1}\left(\frac{\alpha^c + \tau^c}{\alpha^c + y^c}\right)^{\frac{\alpha^c.c-\alpha^c-\tau^c}{\tau^c.c}+1} \tag{14}$$

Using Equations (3) and (14), the IBXIILF is formally defined as:

$$L_{IBXII}(y,\tau) = K\left\{1-\left(\left(\frac{y}{\tau}\right)^{c_L-1}\left(\frac{{\alpha_L}^{c_L} + \tau^{c_L}}{{\alpha_L}^{c_L} + y^{c_L}}\right)^{\frac{{\alpha_L}^{c_L}.c_L-{\alpha_L}^{c_L}-\tau^{c_L}}{\tau^{c_L}.c_L}+1}\right)\right\} \tag{15}$$

Here, τ is the target value, α_L is the scale parameter for the loss function, c_L is the shape parameter for the loss function, and K is the maximum loss. The IBXIILF is represented as IBXIILF(τ, α_L, c_L) providing a flexible and precise tool for modeling loss across symmetric and asymmetric data distributions. To illustrate its behaviour, an example is provided where the distributions of IBXIILF(3,5,5) and Burr XII(5,5,10) are shown in Figure 2 for τ = 3. IBXIILF(3,5,5) is obtained inverted by Burr XII(5,5,10).

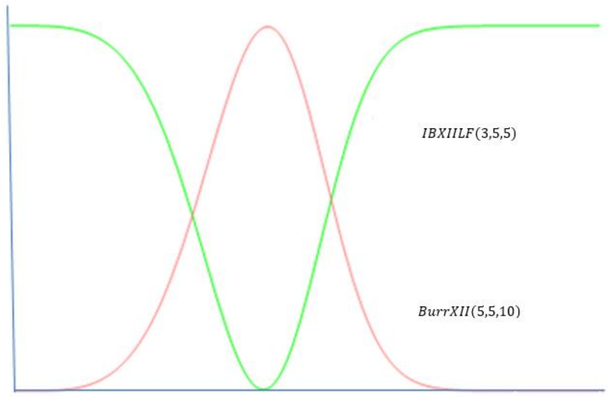

Figure 2. The Burr XII distribution and IBXIILF obtained by its inversion. ($\tau = 3$)

The flexibility of the IBXIILF in handling both symmetric and asymmetric loss processes is demonstrated in Figure 3, where the effects of varying c_L and α_L parameters are shown.

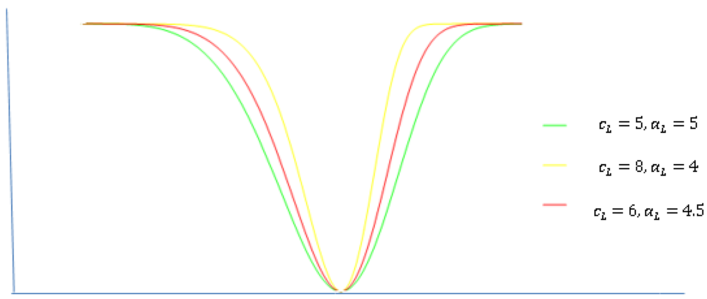

Figure 3. IBXIILF for τ = 3 obtained at different values of c_L and α_L.

Expected Loss Calculation

The expected loss, representing the average economic impact, is calculated as:

The distribution of the quality characteristic and loss of interest, expressed by the Burr XII distribution, is the expected loss.

$$EL_{IBXII} = \int_0^\infty L_{IBXII}(y,\tau) * \mathrm{f(y)}dy \tag{16}$$

$$EL_{IBXII} = \int_0^\infty K\left\{1 - \left(\left(\frac{y}{\tau}\right)^{c_L - 1} \left(\frac{\alpha_L{}^{c_L} + \tau^{c_L}}{\alpha_L{}^{c_L} + y^{c_L}}\right)^{\frac{\alpha_L{}^{c_L}.c_L - \alpha_L{}^{c_L} - \tau^{c_L}}{\tau^{c_L}.c_L} + 1}\right)\right\} \frac{ck}{\alpha}\left(\frac{y}{\alpha}\right)^{c-1}\left(1 + \left(\frac{y}{\alpha}\right)^{c}\right)^{-(k+1)} dy \quad (17)$$

Here, f(y) is the Burr XII pdf, and $L_{IBXII}(y,\tau)$ quantifies the loss for deviations from the target value.

4. COMPARISON OF INVERTED PROBABILISTIC LOSS FUNCTIONS

Different loss functions in the literature yield varying levels of losses for similar deviations from the target value. Leung and Spiring (2004) emphasized that the accuracy of expected loss calculations depends on selecting a loss structure that aligns with the process distribution. This alignment is achieved by using a conjugate distribution for the quality characteristic of the process.

In this section, we compare the performance of four loss functions—INLF, IGLF, IWLF, and IBXIILF—under homogeneous conditions by selecting the Normal, Gamma, Wald, and Burr XII distributions with identical means and variances as the process distributions. The analysis evaluates the impact of variance and target values on the expected losses for these loss functions.

To ensure consistency, the following steps were applied:

1. The variances in the arms of the loss functions were kept identical across all comparisons.
2. The loss functions were evaluated under two distinct scenarios: one with small variance ($\sigma^2 = 0.25$) and another with large variance ($\sigma^2 = 12$).
3. Three levels of variance ($\sigma_L{}^2$=0.5, 0.75, and 1) were used to assess the impact of variance size on loss.
4. Three target values were chosen to examine their effect on losses:
   - Near the lower specification limit ($\tau = 0.5$).
   - At the midpoint of the specification range ($\tau = 1$).
   - Close to the upper specification limit ($\tau = 1.5$).

Table 1. Determined parameter values of loss functions

Distribution \ τ	0.5	1	1.5
Normal (τ, σ^2)	$INLF(0.5, 0.5)$	$INLF(1, 0.75)$	$INLF(1.5, 1)$
Gamma (τ, β)	$IGLF(0.5,2)$	$IGLF(1,3)$	$IGLF(1.5,4)$
Wald (τ, λ)	$IWLF(0.5, 2)$	$IWLF(1, 5.08)$	$IWLF(1.5, 9.22)$
BurrXII (τ,α,c)	*IBXIILF*(0.5,3.12,1.64)	*IBXIILF*(1,2.75,2.17)	*IBXIILF(*1.5,3.26,2.55)

Table 1 summarizes the determined parameter values for each loss function based on the selected process distributions. Tables 2 and 3 detail the process distributions corresponding to these scenarios, highlighting the quality characteristics used for analysis under each variance condition.

Table 2. Homogeneous process distributions

Distribution [$\mu = 1, \sigma^2 = 0.25$]	Distribution [$\mu = 6, \sigma^2 = 12$]	Sample Space
$N(1,0.25)$	$N(6,12)$	$-\infty < x < \infty$
$G(4,0.25)$	$G(3,2)$	$0 < x < \infty$
W(1,4)	W(6,18)	$0 < x < \infty$
BurrXII (1.52,2.61,2.84)	BurrXII (10.33,2.25,3.27)	$0 < x < \infty$

Table 3. Comparison of expected losses when process variability $\sigma^2 = 0.25$

	$\tau = 0.5$	$\sigma_L{}^2 = 0.5$		
Distribution [$\mu = 1, \sigma^2 = 0.25$]	INLF (0.5,0.5)	IGLF (0.5,2)	IWLF(0.5,2)	IBXIILF(0.5,3.12,1.64)
N(1,0.25)	0.30885	0.32019	0.44516	0.34539
G(4,0.25)	0.27268	0.28407	0.39404	0.30735
W (1,4)	0.25801	0.26676	0.36506	0.28936
BurrXII (1.52,2.61,2.84)	0.27497	0.28909	0.40314	0.31224

	$\tau = 1$	$\sigma_L^2 = 0.75$		
Distribution [$\mu = 1, \sigma^2 = 0.25$]	INLF (1,0.75)	IGLF (1,3)	IWLF(1, 5.08)	IBXIILF(1,2.75,2.17)
N(1,0.25)	0.13397	0.1839	0.27502	0.19385
G(4,0.25)	0.12718	0.1891	0.29787	0.19901
W (1,4)	0.11920	0.1707	0.2772	0.18163
BurrXII (1.52,2.61,2.84)	0.12659	0.1886	0.29276	0.19795

	$\tau = 1.5$	$\sigma_L^2 = 1$		
Distribution [$\mu = 1, \sigma^2 = 0.25$]	INLF(1.5,1)	IGLF(1.5,4)	IWLF(1.5, 9.22)	IBXIILF(1.5,3.26,2.55)
N(1,0.25)	0.19069	0.2981	0.41472	0.28612
G (4,0.25)	0.20187	0.3469	0.49017	0.33182
W(1,4)	0.20298	0.3482	0.50530	0.33334
BurrXII (1.526,2.61,2.84)	0.20118	0.3434	0.48402	0.32920

Table 4. Comparison of expected losses when process variability $\sigma^2 = 12$

	$\tau = 0.5$	$\sigma_L^2 = 0.5$		
Distribution [$\mu = 6, \sigma^2 = 12$]	INLF(0.5,0.5)	IGLF(0.5,2)	IWLF(0.5,2)	IBXIILF(0.5,3.11,1.64)
N(6,12)	0.94036	0.9022	0.91574	0.90828
G(3,2)	0.96095	0.9478	0.96132	0.95733
W(6,18)	0.98734	0.9730	0.98123	0.98157
BurrXII (10.33,2.25,3.27)	0.96076	0.9489	0.96168	0.95765

	$\tau = 1$	$\sigma_L^2 = 0.75$		
Distribution [$\mu = 6, \sigma^2 = 12$]	INLF(1,0.75)	IGLF (1,3)	IWLF(1, 5.08)	IBXIILF(1,2.74,2.17)
N(6,12)	0.90901	0.8656	0.87933	0.84204
G(3,2)	0.90612	0.8865	0.90143	0.85101
W(6,18)	0.94333	0.9186	0.93127	0.88035
BurrXII(10.33,2.25,3.27)	0.91130	0.8932	0.90746	0.85999

	$\tau = 1.5$	$\sigma_L^2 = 1$		
Distribution [$\mu = 6, \sigma^2 = 12$]	INLF (1.5,1)	IGLF(1.5,4)	IWLF(1.5, 9.22)	IBXIILF(1.5,3.26,2.55)
N(6,12)	0.87271	0.8259	0.83990	0.84468
G(3,2)	0.83861	0.8172	0.83325	0.84561
W(6,18)	0.86807	0.8383	0.85225	0.87214
BurrXII (10.33,2.25,3.27)	0.84989	0.8295	0.84501	0.85675

The expected losses for various target values, variances, and process distributions of the loss functions are presented in Tables 3 and 4. For small variance ($\sigma^2 = 0.25$), the process distributions are defined as $N(1,0.25), G(4,0.25), W(1,4)$, $BurrXII(1.52,2.61,2.84)$) and for large variance ($\sigma^2 = 12$) , the corresponding distributions are $N(6,12), G(3,2), W(6,18)$ and $BurrXII(10.33,2.25,3.27)$. (Koksoy et. al., 2019).

Leung and Spiring (2004) and Köksoy et al. (2019) emphasized the importance of selecting a loss function that aligns with the quality characteristic distribution to ensure accurate expected loss calculations. This alignment is achieved using conjugate distributions—specific loss functions tailored to the process distribution.

Tables 3 and 4 reveal that different loss functions yield varying expected losses, highlighting the critical role of selecting a suitable loss structure. When the loss function is not aligned with the process distribution, expected losses may be inaccurately measured.

In this paper, IBXIILF is proposed as the conjugate loss function for the Burr XII distribution. This alignment enhances the accuracy of loss measurement, particularly for processes following the Burr XII distribution. These findings underscore the necessity of selecting appropriate loss functions to ensure reliable and precise quality assessments.

5. EXAMPLE: A PRINTING PROCESS STUDY

An experiment on the "printing process," frequently discussed in the literature, was chosen for this study. The **quality characteristic** of interest was "**registration of two images**," defined as the distance between the images along the web. Subgroups of four sheets were sampled regularly, and the distance between images (measured in tenths of an inch) for each sheet was recorded. The data collected is presented in Table 5**.** In this experiment, the target registration is 3, while registrations exceeding 10 are treated as scrap (Leung & Spiring, 2004).

Table 5. Printing Process Data

3.5	2.9	1.4	3.9	6.6	4.1
3.8	6.0	1.9	3.5	6.8	3.8
4.0	6.3	3.0	3.0	6.3	3.7
3.7	6.5	3.9	10.0	6.1	3.6
4.0	6.4	3.5	10.6	5.6	2.6
4.5	4.5	3.3	10.5	5.9	5.0
4.1	4.9	5.5	9.9	4.9	5.9
4.2	4.7	4.0	9.5	4.5	11.9
2.5	1.0	5.0	8.8	4.0	5.0
2.8	1.5	3.6	9.1	4.4	3.4

The measurements from the printing process were analyzed by fitting them to Wald, Gamma, and Burr XII distributions. The goodness-of-fit was evaluated using the Anderson-Darling test, yielding the following p-values: p=0.7345 for the Burr XII distribution, p=0.3744 for the Gamma distribution, and p=0.4087 for the Wald distribution. Among these, the Burr XII distribution demonstrated the best fit to the data. This superior fit suggests that using the Burr XII distribution enhances the accuracy of measurement results.

To further illustrate the fit of these distributions, a histogram, empirical cumulative density functions (CDFs), and quantile-quantile (QQ) plots are provided in Figure 4. While the fitted Wald(4.9967,18.50534.9967, 18.50534.9967,18.5053), Gamma(4.3765,1.10494.3765,1.10494.3765,1.1049), BurrXII(4.6815,3.6709,1.10494.6815, 3.6709, 1.10494.6815,3.6709,1.1049) distributions all show similar overall fits, the Burr XII distribution outperforms the others, particularly in capturing the middle percentiles and the spread of the right tail of the printing process data. This demonstrates the Burr XII distribution's capability to model such data with greater precision, further validating its suitability for use in quality and process control.

Different loss functions offer varying capabilities in measuring loss. When calculating the true process loss, it is crucial to select a loss function that aligns closely with the data distribution. Table 6 presents the goodness-of-fit and expected loss results. These results show that using a loss structure mismatched to the data distribution leads to inaccuracies in expected loss calculations. The more the distribution deviates from the actual data, the more unreliable the loss measurement becomes.

Table 6. Expected losses for the printing process test

	IGLF(3, 4.8533)	***A.D. p-value**
Gamma(4.3765, 1.1049)	4.2782	0.3744
	IBXIILF(3, 3.8217, 3.7710)	
BurrXII(4.6815, 3.6709,1.1049)	5.2640	0.7345
	IWLF(3, 18.4348)	
Wald(4.9967, 18.5053)	4.1322	0.4087

The Burr XII distribution, which fits the data best, allows more accurate loss measurement when paired with its conjugate loss function, IBXIILF. Equation (17) is used to calculate the expected loss with IBXIILF, and the expected values of IGLF and IWLF are derived from Equations (5) and (7), respectively.

For the Burr XII (4.6815,3.6709,1.1049, the IBXIILF is expressed as:

$$EL_{IBXII}(y,\tau) = 10\left\{1-\left(\left(\frac{y}{3}\right)^{3.7710-1}\left(\frac{3.8217^{3.7710}+3^{3.7710}}{3.8217^{3.7710}+y^{3.7710}}\right)^{\frac{3.8217^{3.7710}.3.7710-3.8217^{3.7710}-3^{3.7710}}{3^{3.7710}.3.7710}+1}\right)\right\} \quad (18)$$

Here, $K = 10$, τ =3, $\alpha_L = 3.8217, c_L = 3.7710$. Using Equation (16), the expected loss is calculated as **5.264 scraps**.

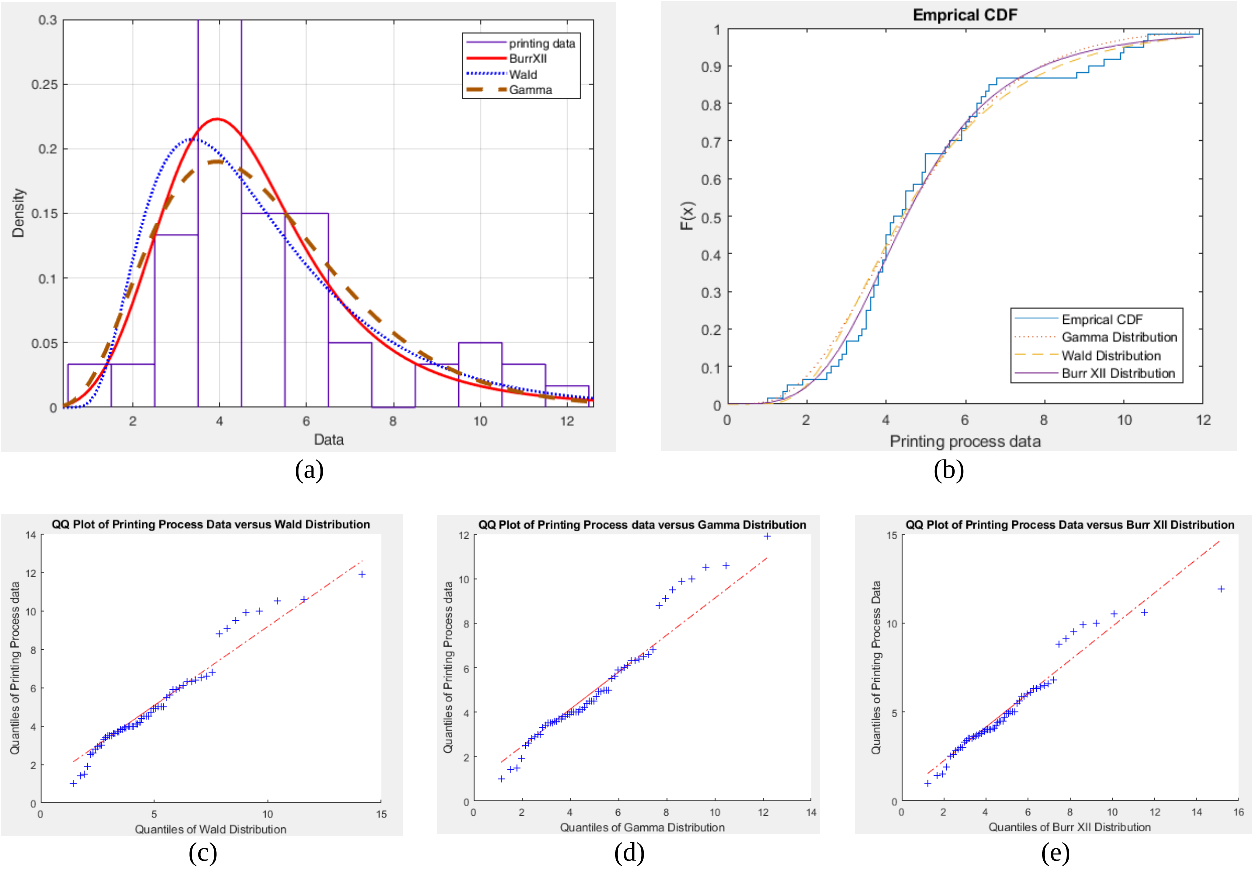

Figure 4. Goodness-of-Fit Analysis for Printing Process Data: (a) Histogram for Wald, Gamma, and Burr XII distributions. (b) Empirical cumulative density functions (CDFs) for the three distributions. (c) Quantile-quantile (QQ) plot for Wald distribution. (d) QQ plot for Gamma distribution. (e) QQ plot for Burr XII distribution.

6. CONCLUSION

In recent years, loss functions have gained significant attention among statisticians and quality engineers due to their emphasis on achieving targets with minimal variation, thereby reducing costs for both customers and manufacturers. This study introduces the Inverted Burr XII Loss Function (IBXIILF) as a novel addition to the Inverted Probability Loss Function (IPLF) family. The IBXIILF is versatile, accommodating right-skewed, left-skewed, or symmetric structures, and provides more realistic loss assessments regardless of the process distribution.

To demonstrate its practical application, a loss-monitoring process was analyzed using the proposed loss function, showcasing its effectiveness in accurately assessing process performance. The IBXIILF offers practitioners a robust tool for evaluating loss in various industrial scenarios and expands the options available for measuring process performance in terms of loss. This study serves as a valuable reference for professionals seeking reliable and adaptable methods for quality and risk management.

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