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A MP-BASED ALGORITHM FOR A MULTICOMMODITY STOCHASTIC-FLOW NETWORK WITH CAPACITY WEIGHTS

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This paper proposes an efficient algorithm to evaluate system reliability for many real-life systems such as manufacturing, telecommunication and computers systems. A multicommodity stochastic-flow network is constructed to model a manufacturing system in which each node stands for a machine station, and each arc stands for a transmission medium (shipping machine or conveyor). Four characteristics are considered: 1) both nodes and arcs have multiple possible capacities and may fail; 2) each component (arc/node) has both capacity and cost attributes; 3) multicommodity are proceed; and 4) the capacity weight varies with arcs, nodes, and types of commodity. We study the possibility that multicommodity can be transmitted through this network simultaneously under the budget constraint. Such a possibility is named the system reliability. The MP (minimal path) plays the role of describing the relationship among flow vectors and capacity vectors. Subsequently, an efficient algorithm in terms of MP is proposed to evaluate the system reliability.

Keywords: Stochastic-flow network, capacity weight, multicommodity, minimal paths, system reliability, cost attribute.

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1. INTRODUCTION

Network analysis is a crucial tool used to solve system capacity problems. In a binary-state network without flow through it, Abraham (1979), Yarlagadda and Hershey (1991) evaluated the system reliability, the probability that source s communicates with sink t, by applying the disjoint event method in terms of minimal paths (MP), where a MP is a path whose proper subsets are no longer paths. Note that a MP is different from the so-called minimum path that is a path with minimum total length. Aggarwal et al. (1975) proposed a concept that the failure of a node implies the failure of arcs incident from it. Then the original network with unreliable nodes can be modified to a conventional network with perfect nodes. Extending to a binary-state flow network, each arc's capacity (the maximum flow passing the arc per unit time) has two values, 0 and a positive integer. System capacity is the maximum flow from s to t. Thus, system reliability is the probability that system capacity is no less than the demand. The evaluation of network reliability had been shown to be NP-hard (Colbourn, 1987). Aggarwal et al. (1982) solved such a reliability problem in terms of MP.

In a stochastic-flow network composed of multistate arcs, the system capacity is not a fixed number. Therefore, such a network is also multistate. A manufacturing system can be also treated as a stochastic-flow network in which each node stands for a machine station and each arc stands for a transmission medium (shipping machine or conveyor). In fact, every machine station is combined with several machines, and each machine has either normal or failure state. Each machine station has several states, where state k denotes that k machines are normal. Hence, the capacity of each node has several values. Without cost attributes, several authors (Xue, 1985; Yeh, 1998) presented algorithms to generate all lower boundary points for demand d in terms of MP. The lower boundary point for d is a minimal system state meeting the demand d, equivalently, it is a minimal system state such that system capacity equals d. The literatures (Lin, 2001[b]; Yeh, 2001) and (Lin, 1998; Yeh, 2005) extended such a reliability problem to unreliable nodes case, and budget constraint case, respectively.

The max-flow min-cut Theorem (Ford and Fulkerson, 1962) states that the maximum flow from s to t equals the minimum capacity among all minimal cuts (MC) where a MC is a cut whose proper subsets are no longer cuts. The Theorem indicates that MP and MC are two important approaches to solve network problems. Hence, the authors (Jane, et al., 1993; Lin, 2001[a]; Soh and Rai, 2005) utilized MC to generate all upper boundary points for d for evaluating system unreliability, the probability that the upper bound of system capacity equals d, for perfect nodes and single-commodity case. An upper boundary point for d is a maximal capacity vector exactly meeting the demand d.

Moreover, multicommodity, multiple types of commodity, are produced thorough a real-life manufacturing network which is called a multicommodity stochastic-flow network. In the past few decades, several authors (Hu, 1963; Rothechild and Whinston, 1966; Jarvis, 1969; Ford and Fulkerson, 1974; Held, et al., 1974; Assad, 1978) solved multicommodity maximal flow problem to find maximal total flow by assuming the arc is deterministic. The total flow, however, is not appropriate to be treated as system capacity, especially in case that different commodity consumes arc capacity differently.

For example, as the data shown in Table 1, the total flow in network A is larger than that in network B. Yet, it does not imply that network A has the better transmission ability when commodity 2 consumes more capacity than commodity 1 does.

Table 1. The total flow for two networks

	Network A	Network B	Capacity weight
Flow of commodity 1	10	5	2
Flow of commodity 2	6	10	3
Sum of the commodity	16	15	
Total capacity consumed	38 (10×2 + 6×3)	40 (5×2 + 10×3)	

A multicommodity stochastic-flow network is constructed with four characteristics: 1) both nodes and arcs have several possible capacities; 2) each component (arc/node) has both capacity and cost attributes; 3) multicommodity are proceed; and 4) the capacity weight varies with arcs, nodes, and types of commodity. The focused problem is how to evaluate system reliability that the system fulfills the multicommodity demand under the budget constraint. For convenience, we first concentrate on a Two-commodity Stochastic-Flow Network (TSFN). The remainder of this work is organized as follows. In section 2, we discuss the relationship among flow vectors and capacity vectors in terms of MP. System capacity and lower boundary points for $(d_1,d_2;B)$ are also defined. System reliability, the probability that the system meets the demand (d_1,d_2) under budget B, can be computed in terms of all lower boundary points for $(d_1,d_2;B)$. In section 3, an efficient algorithm based on MP is proposed to generate all lower boundary points for $(d_1,d_2;B)$. A benchmark example is shown in Section 4 to illustrate the proposed algorithm and how system reliability may be computed. The storage and computational time complexity are analyzed in Section 6.

2. ASSUMPTIONS AND NOMENCLATURE

Let $G(A, Q, M, C, W)$ be a TSFN where $A = \{a_i|1 \le i \le n\}$ is the set of arcs, $Q = \{a_i|n + 1 \le i \le n + q\}$ is the set of nodes, $M \equiv (M_1, M_2, \ldots, M_{n+q})$ with M_i being the maximal capacity of a_i, $C = \{ c_i^k \mid i = 1, 2, \ldots, n + q, k = 1, 2\}$ with c_i^k being the cost through a_i per commodity k and $W = \{ w_i^k \mid i = 1, 2, \ldots, n + q, k = 1, 2\}$ with w_i^k being the capacity weight and denoting the consumed capacity on a_i per commodity k. The current capacity of component a_i is denoted by x_i, and the vector $X \equiv (x_1, x_2, \ldots, x_{n+q})$ is called a capacity vector representing the system state. Without loss of generality, we assume that $w_i^2 \ge w_i^1 \ge 1$ for each component a_i. The network G is required to satisfy the following assumptions.

1. Two types of commodity are transmitted from s to t.
2. The current capacity x_i takes values from $\{0, 1, 2, \ldots, M_i\}$, $i = 1, 2, \ldots n + q$.
3. Capacities of different components are statistically independent.
4. Flow of each type of commodity must satisfy the flow conservation (Ford and Fulkerson, 1962).
5. Both source and sink have infinite capacity, and are perfect.

2.1 Nomenclature

$\lceil x \rceil$ the smallest integer such that $\lceil x \rceil \ge x$

$(d_1,d_2) + (d_1',d_2')$: $(d_1 + d_1',d_2 + d_2')$

Vector comparisons are made as follows.

$X \le Y$ $(x_1, x_2, \ldots, x_{n+q}) \le (y_1, y_2, \ldots, y_{n+q})$: $x_i \le y_i$ for $i = 1, 2, \ldots, n + q$

$X < Y$ $(x_1, x_2, \ldots, x_{n+q}) < (y_1, y_2, \ldots, y_{n+q})$: $X \le Y$ and $x_i < y_i$ for at least one i

$(d_1,d_2) \le (d_1',d_2')$: $d_k \le d_k'$ for $k = 1, 2$

$(d_1,d_2) < (d_1',d_2')$: $(d_1,d_2) \le (d_1',d_2')$ & $d_k < d_k'$ for at least one k

3. TWO-COMMODITY STOCHASTIC-FLOW NETWORKS

Since the nodes are unreliable, we redefine a path as an ordered sequence of arcs and nodes that connects s and t. MP is also redefined as an ordered sequence of arcs and nodes whose proper subsets are no longer paths. Suppose $P_1, P_2, \ldots, P_m$ are MPs. The TSFN is described in terms of capacity vector $X = (x_1, x_2, \ldots, x_{n+q})$ and flow vector (F^1,F^2) where $F^1 = (f_1^1, f_2^1, \ldots, f_m^1)$ and $F^2 = (f_1^2, f_2^2, \ldots, f_m^2)$ with f_j^k denoting flow of commodity k through P_j, $j = 1, 2, \ldots, m$, $k = 1, 2$. Such a flow vector is feasible under X if it satisfies the following condition,

$$\left[w_i^1 \sum_{a_i \in P_j} f_j^1 + w_i^2 \sum_{a_i \in P_j} f_j^2 \right] \le x_i \text{ for } i = 1, 2, ..., n+q. \qquad \cdots \quad (1)$$

Inequality (1) says that the total quantity $\left[w_i^1 \sum_{a_i \in P_j} f_j^1 + w_i^2 \sum_{a_i \in P_j} f_j^2 \right]$ of capacity on a_i consumed by (F^1,F^2) cannot exceed the current capacity x_i. For convenience, let ϕ_X denote the set of (F^1,F^2) feasible under X. Similarly, ϕ_M is the set of all flow vectors.

The flow vector (F^1,F^2) is said to meet both demand and budget constraints if it satisfies constraints (2) – (4),

$$\sum_{j=1}^{m} f_j^k = d_k, k = 1, 2, \qquad \cdots \quad (2)$$

$$\sum_{i=1}^{n} \left(c_i^1 \sum_{a_i \in P_j} f_j^1 + c_i^2 \sum_{a_i \in P_j} f_j^2 \right) \le B, \qquad \cdots \quad (3)$$

$$\left[w_i^1 \sum_{a_i \in P_j} f_j^1 + w_i^2 \sum_{a_i \in P_j} f_j^2 \right] \le M_i \text{ for } i = 1, 2, ..., n+q. \qquad \cdots \quad (4)$$

The value $\sum_{i=1}^{n} \left(c_i^1 \sum_{a_i \in P_j} f_j^1 + c_i^2 \sum_{a_i \in P_j} f_j^2 \right)$ is the total transmission cost under (F^1,F^2). Hence, Equation (2) means that (F^1,F^2) meets the demand constraint, constraint (3) means that (F^1,F^2) meets the budget constraint, and constraint (4) indicates that $(F^1,F^2) \in \phi_M$. For convenience, let Φ be the set of all flow vectors meeting both demand and budget constraints.

3.1 System reliability evaluation

System capacity $V(X)$ under X is defined as (d_1, d_2) if at most (d_1, d_2) can be proceed through the TSFN. Equivalently, system capacity $V(X)$ is (d_1, d_2) if any demand (d_1', d_2') with $(d_1', d_2') > (d_1, d_2)$ cannot be proceed through the TSFN. Different from the single-commodity case, system capacity is not certainly unique. For instance in Figure 1, there are 2 MPs: $P_1 = \{a_7, a_1, a_5, a_2, a_8\}$ and $P_2 = \{a_7, a_3, a_6, a_4, a_8\}$. Set $w_i^1 = 1$, $w_i^2 = 3$ and $M_i = 4$ for all components a_i. System capacity $V(X)$ is (2,2) if $(f_1^1, f_2^1, f_1^2, f_2^2) = (1,1,1,1)$, and also (5,1) if $(f_1^1, f_2^1, f_1^2, f_2^2) = (1,1,4,0)$.

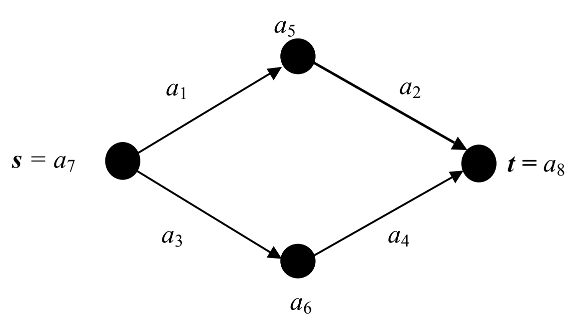

Figure 1. A network

The capacity vector X is said to meet both demand (d_1, d_2) and budget constraints if there exists a $(F^1,F^2) \in \phi_X$ meeting both demand and budget constraints. Let Ω be the set of such X. System reliability $R_{d_1,d_2;B}$ is thus

$R_{d_1,d_2;B}$ = Pr{X|X meets both demand and budget constraints}

= Pr{X|$V(X) \geq (d_1, d_2)$ and there exists a $(F^1,F^2) \in \phi_X$ such that $\sum_{i=1}^{n}\left(c_i^1 \sum_{a_i \in P_j} f_j^1 + c_i^2 \sum_{a_i \in P_j} f_j^2\right) \leq B$ }

$$= \Pr\{\Omega\} = \sum_{X\in\Omega} \Pr\{X\}, \quad \cdots \quad (5)$$

where $\Pr\{X\} = \Pr\{x_1\} \times \Pr\{x_2\} \times \ldots \times \Pr\{x_{n+q}\}$ by assumption 3 (note that $\Pr\{x_i\}$ is the probability that the capacity of a_i is exact x_i). However, a straightforward method to enumerate all X in Ω is not a wise way if the network size is large. It will be more efficient to evaluate $R_{d_1,d_2;B}$ if the minimal capacity vectors in Ω can be found in advance.

Definition: The minimal vector in Ω is defined as a lower boundary point for $(d_1,d_2;B)$. Equivalently, X is a lower boundary point for $(d_1,d_2;B)$ if and only if i) $X \in \Omega$, and ii) $Y \notin \Omega$ for any capacity vector Y such that $Y < X$.

Suppose there are r lower boundary points for $(d_1,d_2;B)$: $X_1, X_2, \ldots, X_r$. Let subset $S_i \equiv \{X|X \geq X_i\}$, $i = 1, 2, \ldots, r$. System reliability can be formulated as follows:

$$R_{d_1,d_2;B} = \Pr\{X|X \geq X_i \text{ for a lower boundary point } X_i \text{ for } (d_1,d_2;B)\} \quad \cdots \quad (6)$$
$$= \Pr\{S_1 \cup S_2 \cup \ldots \cup S_r\}.$$

It can be calculated by applying several methods such as inclusion-exclusion principal (Griffith, 1980; Hudson and Kapur, 1985; Lin, 2001[a], 2001[b], 2003; Yeh, 2004, 2005; Lin, 2006, 2007[a], 2007[b]), disjoint subsets (Xue, 1985; Lin, 2001[a]) and state-space decomposition (Jane, et al., 1993; Lin, 1998).

3.2 Generate all lower boundary points for $(d_1,d_2;B)$

The following Theorem shows a necessary condition for a lower boundary point for $(d_1,d_2;B)$.

Theorem 1. Let X be a lower boundary point for $(d_1,d_2;B)$. Then there exists an $(F^1,F^2) \in \Phi$ such that.

$$x_i = \left\lceil w_i^1 \sum_{a_i \in P_j} f_j^1 + w_i^2 \sum_{a_i \in P_j} f_j^2 \right\rceil \text{ for } i = 1, 2, ..., n+q. \quad \cdots \quad (7)$$

Proof: For each $(F^1,F^2) \in \Phi$, constraint (1) have stated that $\left\lceil w_i^1 \sum_{a_i \in P_j} f_j^1 + w_i^2 \sum_{a_i \in P_j} f_j^2 \right\rceil \leq x_i$ for $i = 1, 2, ..., n+q$. Without loss of generality, suppose $\left\lceil w_1^1 \sum_{a_1 \in P_j} f_j^1 + w_1^2 \sum_{a_1 \in P_j} f_j^2 \right\rceil < x_1$, then (F^1,F^2) is feasible under the capacity vector $(X + e_1)$ where e_1 is a $(n+q)$-tuple vector with 1 at position 1 and 0 at others. It means that $(X + e_1) \in \Omega$ which contradicts that X is minimal in Ω. Hence, $\left\lceil w_i^1 \sum_{a_i \in P_j} f_j^1 + w_i^2 \sum_{a_i \in P_j} f_j^2 \right\rceil = x_i$ for $i = 1, 2, ..., n+q$.

Q.E.D.

For each $(F^1,F^2) \in \Phi$, generate the corresponding capacity vector $Z_{F^1,F^2} = (z_1, z_2, \ldots, z_{n+q})$ via $z_i = \left\lceil w_i^1 \sum_{a_i \in P_j} f_j^1 + w_i^2 \sum_{a_i \in P_j} f_j^2 \right\rceil$ for $i = 1, 2, ..., n+q$. In fact, Z_{F^1,F^2} meets both demand and budget constraints because (F^1,F^2) is feasible under Z_{F^1,F^2}. We call such Z_{F^1,F^2} a candidate of lower boundary point for $(d_1,d_2;B)$. For convenience, let $\Psi = \{Z_{F^1,F^2} | (F^1,F^2) \in \Phi\}$ be the set of all candidates of lower boundary point for $(d_1,d_2;B)$. The following Theorem further shows that $\Psi_{\min} \equiv \{X|X$ is minimal in $\Psi\}$ is the set of lower boundary points for $(d_1,d_2;B)$.

Theorem 2. $\{X|X$ is a lower boundary point for $(d_1,d_2;B)\} = \Psi_{\min}$.

Proof: Firstly, suppose that X is a lower boundary point for $(d_1,d_2;B)$ (note that $X \in \Psi$) but $X \notin \Psi_{\min}$ i.e., there exist a $Y \in \Psi$ such that $Y < X$. Then $Y \in \Omega$, which contradicts that X is a lower boundary points for $(d_1,d_2;B)$. Hence, $X \in \Psi_{\min}$.

Conversely, suppose that $X \in \Psi_{\min}$ (note that $X \in \Omega$) but it is not a lower boundary point for $(d_1,d_2;B)$. Then there exists a lower boundary point Y for $(d_1,d_2;B)$ such that $Y < X$. Therefore, $Y \in \Psi$ that contradicts that $X \in \Psi_{\min}$. Hence, X is a lower boundary point for $(d_1,d_2;B)$. **Q.E.D.**

4. ALGORITHM

Similar to those algorithms (Xue, 1985; Lin, 1998; Yeh, 1998; Lin, 2001[b]; Yeh, 2001; Lin, 2003; Yeh, 2005; Lin, 2007[a]), we suppose all MP have been pre-computed. Virtually, MP can be efficiently derived from those algorithms discussed in (Shen, 1995; Al-Ghanim, 1999; Kobayashi and Yamamoto, 1999). The algorithm proposed by Al-Ghanim (1999) showed an approximate linear time response versus the number of nodes. All lower boundary points for $(d_1,d_2;B)$ can be generated as follows.

Step 1. $v \leftarrow 0$, $\mathrm{I} \leftarrow \phi$, $\Psi \leftarrow \phi$, $\Psi_{\max} \leftarrow \phi$.

Step 2. Obtain all (F^1,F^2) satisfying demand, budget, and capacity constraints.

$$\sum_{j=1}^{m} f_j^k = d^k,\ k = 1, 2, \qquad \cdots \qquad (8)$$

$$\sum_{i=1}^{n}\left(c_i^1 \sum_{a_i \in P_j} f_j^1 + c_i^2 \sum_{a_i \in P_j} f_j^2\right) \le B, \qquad \cdots \qquad (9)$$

$$\left\lceil w_i^1 \sum_{a_i \in P_j} f_j^1 + w_i^2 \sum_{a_i \in P_j} f_j^2 \right\rceil \le M_i \text{ for } i = 1, 2, ..., n + q. \qquad \cdots \qquad (10)$$

Step 3. Transform each (F^1,F^2) into $X = (x_1, x_2, \ldots, x_{n+q})$ via

$$x_i = \left\lceil w_i^1 \sum_{a_i \in P_j} f_j^1 + w_i^2 \sum_{a_i \in P_j} f_j^2 \right\rceil \text{ for } i = 1, 2, ..., n + q. \qquad \cdots \qquad (11)$$

Set $v \leftarrow v + 1$, $X_v \leftarrow X$, and $\Psi \leftarrow \Psi \cup X_v$.

Step 4. $\Psi = \{X_1, X_2, \ldots, X_v\}$.

4.1) **for** $i = 1$ **to** v with $i \notin \mathrm{I}$
4.2) **for** $j = i + 1$ **to** v with $j \notin \mathrm{I}$
4.3) **if** $X_i \ge X_j$, $\mathrm{I} = \mathrm{I} \cup \{i\}$ and go to step 4.6)
elseif $X_j > X_i$, $\mathrm{I} = \mathrm{I} \cup \{j\}$
4.4) $j = j + 1$
4.5) X_i is a lower boundary point for $(d_1,d_2;B)$, and $\Psi_{\min} \leftarrow \Psi_{\min} \cup X_i$
4.6) $i = i + 1$
4.7) **end**

Equation (11) guarantees that X is a candidate of lower boundary points for $(d_1,d_2;B)$. Use the comparison method to store the lower boundary points for $(d_1,d_2;B)$ into $\Psi_{\min}$. I is used to store the index of X which is not a lower boundary point for $(d_1,d_2;B)$. If $X_i \ge X_j$, then $i \in \mathrm{I}$. If $X_j > X_i$, then $j \in \mathrm{I}$.

5. AN APPLICATION IN MANUFACTURING SYSTEMS

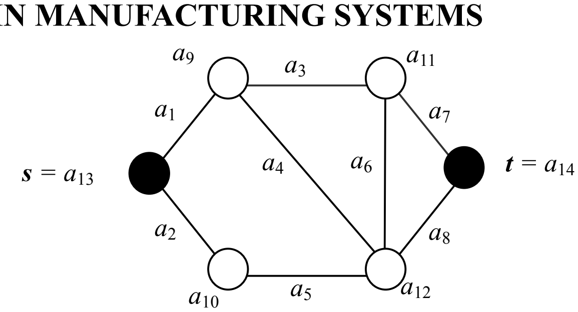

Figure 2. A Benchmark (Soh and Rai, 1991; Yarlagadda and Hershey, 1991)

Table 2. The data of arcs and nodes

Component	Capacity	Probability	w_i^1	w_i^2	c_i^1 **	c_i^2
a_1	0*	.01	1	2	30	60
	1	.01				
	2	.01				
	3	.01				
	4	.02				
	5	.94				
a_2	0	.01	1	2	60	90
	1	.01				
	2	.01				
	3	.02				
	4	.02				
	5	.93				
a_3	0	.01	1	2	90	120
	1	.01				
	2	.01				
	3	.01				
	4	.01				
	5	.95				
a_4	0	.01	1	2	60	120
	1	.01				
	2	.02				
	3	.03				
	4	.03				
	5	.90				
a_5	0	.01	1	2	30	60
	1	.01				
	2	.01				
	3	.01				
	4	.02				
	5	.94				
a_6	0	.01	1	2	60	90
	1	.01				
	2	.02				
	3	.02				
	4	.03				
	5	.91				
a_7	0	.01	1	2	90	120
	1	.01				
	2	.01				
	3	.01				
	4	.01				
	5	.95				
a_8	0	.01	1	2	60	120
	1	.01				
	2	.01				
	3	.03				
	4	.03				
	5	.91				
$a_9 \sim a_{12}$	0	.01	1	2	60	90
	1	.01				
	3	.01				
	5	.01				
	7	.02				
	9	.94				

*Pr{the capacity of a_1 is 0} = 0.01.
**US dollars per lot

We use the benchmark (Soh and Rai, 1991; Yarlagadda and Hershey, 1991) (see Figure 2) modeling a manufacturing system to illustrate the proposed resolution procedure. There are 7 MPs: $P_1 = \{a_{13}, a_1, a_9, a_3, a_{11}, a_7, a_{14}\}$, $P_2 = \{a_{13}, a_1, a_9, a_3, a_{11}, a_6, a_{12}, a_8, a_{14}\}$, $P_3 = \{a_{13}, a_1, a_9, a_4, a_{12}, a_8, a_{14}\}$, $P_4 = \{a_{13}, a_1, a_9, a_4, a_{12}, a_6, a_{11}, a_7, a_{14}\}$, $P_5 = \{a_{13}, a_2, a_{10}, a_5, a_{12}, a_8,$

a_{14}}, P_6 = {a_{13}, a_2, a_{10}, a_5, a_{12}, a_6, a_{11}, a_7, a_{14}}, and P_7 = {a_{13}, a_2, a_{10}, a_5, a_{12}, a_4, a_9, a_3, a_{11}, a_7, a_{14}}. The data of arcs and nodes are shown in Table 2. One lot means 100 homogeneous commodities. Component capacity is measured in terms of hours. For example, $X_{11} = 3$ denotes that station 11 has 3 hour-capacity in one hour, and $w_{11}^{2} = 2$ denotes that one lot of commodity 2 consumes 2 hour-capacity through station 11. If demand (d_1, d_2) is set to be (3 lots, 3 lots) and C = \$2450, the manager would like to know network reliability $R_{3,3;2450}$. First, all lower boundary points for (3,3;2450) can be found by the following steps.

Step 1. $v=0$, $\mathrm{I} = \phi$, $\Psi = \phi$, $\Psi_{\max} = \phi$.

Step 2. Obtain all flow vectors (F^1,F^2) with $F^1 = (f_1^1, f_2^1, f_3^1, f_4^1, f_5^1, f_6^1, f_7^1)$ and $F^2 = (f_1^2, f_2^2, f_3^2, f_4^2, f_5^2, f_6^2, f_7^2)$ satisfying all demand constraint (12), budget constraint (13), and capacity constraint (14),

$$f_1^1 + f_2^1 + f_3^1 + f_4^1 + f_5^1 + f_6^1 + f_7^1 = 3, \quad \cdots \quad (12)$$
$$f_1^2 + f_2^2 + f_3^2 + f_4^2 + f_5^2 + f_6^2 + f_7^2 = 3,$$

$$\{30(f_1^1 + f_2^1 + f_3^1 + f_4^1)+60(f_1^2 + f_2^2 + f_3^2 + f_4^2)\} +$$
$$\{60(f_5^1 + f_6^1 + f_7^1) + 90(f_5^2 + f_6^2 + f_7^2)\} + \{90(f_1^1 + f_2^1 + f_7^1) + 120(f_1^2 + f_2^2 + f_7^2)\} + \{60(f_3^1 + f_4^1 + f_7^1) +$$
$$120(f_3^2 + f_4^2 + f_7^2)\} + \{30(f_5^1 + f_6^1 + f_7^1) + 60(f_5^2 + f_6^2 + f_7^2)\} + \{60(f_2^1 + f_4^1 + f_6^1) + 90(f_2^2 + f_4^2 + f_6^2)\} +$$
$$\{90(f_1^1 + f_4^1 + f_6^1 + f_7^1) +$$
$$120(f_1^2 + f_4^2 + f_6^2 + f_7^2)\} + \{60(f_2^1 + f_3^1 + f_5^1) + 120(f_2^2 + f_3^2 + f_5^2)\} +$$
$$\{60(f_1^1 + f_2^1 + f_3^1 + f_4^1 + f_7^1) + 90(f_1^2 + f_2^2 + f_3^2 + f_4^2 + f_7^2)\} +$$
$$\{60(f_5^1 + f_6^1 + f_7^1) + 90(f_5^2 + f_6^2 + f_7^2)\} + \{60(f_1^1 + f_2^1 + f_4^1 + f_6^1 + f_7^1) + 90(f_1^2 + f_2^2 + f_4^2 + f_6^2 + f_7^2)\} +$$
$$\{60(f_2^1 + f_3^1 + \ldots + f_7^1) +$$
$$90(f_2^2 + f_3^2 + \ldots + f_7^2)\} \le 2450, \quad \cdots \quad (13)$$

$$a_1: \lceil f_1^1 + f_2^1 + f_3^1 + f_4^1 + 2f_1^2 + 2f_2^2 + 2f_3^2 + 2f_4^2 \rceil \le 5, \quad \cdots \quad (14)$$
$$a_2: \lceil f_5^1 + f_6^1 + f_7^1 + 2f_5^2 + 2f_6^2 + 2f_7^2 \rceil \le 5,$$
$$a_3: \lceil f_1^1 + f_2^1 + f_7^1 + 2f_1^2 + 2f_2^2 + 2f_7^2 \rceil \le 5,$$
$$a_4: \lceil f_3^1 + f_4^1 + f_7^1 + 2f_3^2 + 2f_4^2 + 2f_7^2 \rceil \le 5,$$
$$a_5: \lceil f_5^1 + f_6^1 + f_7^1 + 2f_5^2 + 2f_6^2 + 2f_7^2 \rceil \le 5,$$
$$a_6: \lceil f_2^1 + f_4^1 + f_6^1 + 2f_2^2 + 2f_4^2 + 2f_6^2 \rceil \le 5,$$
$$a_7: \lceil f_1^1 + f_4^1 + f_6^1 + f_7^1 + 2f_1^2 + 2f_4^2 + 2f_6^2 + 2f_7^2 \rceil \le 5,$$
$$a_8: \lceil f_2^1 + f_3^1 + f_5^1 + 2f_2^2 + 2f_3^2 + 2f_5^2 \rceil \le 5,$$
$$a_9: \lceil f_1^1 + f_2^1 + f_3^1 + f_4^1 + f_7^1 + 2f_1^2 + 2f_2^2 + 2f_3^2 + 2f_4^2 + 2f_7^2 \rceil \le 9,$$
$$a_{10}: \lceil f_5^1 + f_6^1 + f_7^1 + 2f_5^2 + 2f_6^2 + 2f_7^2 \rceil \le 9,$$
$$a_{11}: \lceil f_1^1 + f_2^1 + f_4^1 + f_6^1 + f_7^1 + 2f_1^2 + 2f_2^2 + 2f_4^2 + 2f_6^2 + 2f_7^2 \rceil \le 9,$$
$$a_{12}: \lceil f_2^1 + f_3^1 + \ldots + f_7^1 + 2f_2^2 + 2f_3^2 + \ldots + 2f_7^2 \rceil \le 9.$$

Seven flow vectors (F^1,F^2) are obtained: (3, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 2, 0, 0), (2, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 2, 0, 0), (2, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 2, 0, 0), (1, 0, 0, 0, 2, 0, 0, 2, 0, 0, 0, 1, 0, 0), (0, 0, 1, 0, 2, 0, 0, 2, 0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 3, 0, 0, 2, 0, 0, 0, 1, 0, 0) and (0, 0, 0, 0, 2, 1, 0, 2, 0, 0, 0, 1, 0, 0). The corresponding costs are 2370, 2370, 2340, 2340, 2340, 2310 and 2430, respectively.

Step 3. For each (F^1,F^2), generate the capacity vector $X = (x_1, x_2, \ldots, x_{12})$ via

$$
\begin{aligned}
x_1 &= \lceil f_1^1 + f_2^1 + f_3^1 + f_4^1 + 2f_1^2 + 2f_2^2 + 2f_3^2 + 2f_4^2 \rceil,\\
x_2 &= \lceil f_5^1 + f_6^1 + f_7^1 + 2f_5^2 + 2f_6^2 + 2f_7^2 \rceil,\\
x_3 &= \lceil f_1^1 + f_2^1 + f_7^1 + 2f_1^2 + 2f_2^2 + 2f_7^2 \rceil,\\
x_4 &= \lceil f_3^1 + f_4^1 + f_7^1 + 2f_3^2 + 2f_4^2 + 2f_7^2 \rceil,\\
x_5 &= \lceil f_5^1 + f_6^1 + f_7^1 + 2f_5^2 + 2f_6^2 + 2f_7^2 \rceil,\\
x_6 &= \lceil f_2^1 + f_4^1 + f_6^1 + 2f_2^2 + 2f_4^2 + 2f_6^2 \rceil, \qquad \cdots \qquad (15)\\
x_7 &= \lceil f_1^1 + f_4^1 + f_6^1 + f_7^1 + 2f_1^2 + 2f_4^2 + 2f_6^2 + 2f_7^2 \rceil,\\
x_8 &= \lceil f_2^1 + f_3^1 + f_5^1 + 2f_2^2 + 2f_3^2 + 2f_5^2 \rceil,\\
x_9 &= \lceil f_1^1 + f_2^1 + f_3^1 + f_4^1 + f_7^1 + 2f_1^2 + 2f_2^2 + 2f_3^2 + 2f_4^2 + 2f_7^2 \rceil,\\
x_{10} &= \lceil f_5^1 + f_6^1 + f_7^1 + 2f_5^2 + 2f_6^2 + 2f_7^2 \rceil,\\
x_{11} &= \lceil f_1^1 + f_2^1 + f_4^1 + f_6^1 + f_7^1 + 2f_1^2 + 2f_2^2 + 2f_4^2 + 2f_6^2 + 2f_7^2 \rceil,\\
x_{12} &= \lceil f_2^1 + f_3^1 + \ldots + f_7^1 + 2f_2^2 + 2f_3^2 + \ldots + 2f_7^2 \rceil.
\end{aligned}
$$

In fact, each value x_i has been calculated by constraint (14) in step 2. We obtain $X_1 = (5, 4, 5, 0, 4, 0, 5, 4, 5, 4, 5, 4)$, $X_2 = (5, 4, 4, 1, 4, 0, 4, 5, 5, 4, 4, 5)$, $X_3 = (4, 5, 4, 0, 5, 0, 4, 5, 4, 5, 4, 5)$, $X_4 = (5, 4, 5, 0, 4, 0, 5, 4, 5, 4, 5, 4)$, $X_5 = (5, 4, 4, 1, 4, 0, 4, 5, 5, 4, 4, 5)$, $X_6 = (4, 5, 4, 0, 5, 0, 4, 5, 4, 5, 4, 5)$ and $X_7 = (4, 5, 4, 0, 5, 1, 5, 4, 4, 5, 5, 5)$. So, $\Psi = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$. In total, we have $\Psi = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$.

Step 4. Check each $X_i \in \Psi$ whether it is a lower boundary point for (3,3; 2450) or not. According to vectors comparisons, store X_i into Ψ_{max} if it is the minimal candidate. Otherwise, delete X_i and store the index i into I.

4.1) $i = 1$
4.2) $j = 2$
4.3) $X_1 \ngeq X_2$ and $X_2 \ngtr X_1$. I = ϕ.
4.2) $j = 3$
4.3) $X_1 \ngeq X_3$ and $X_3 \ngtr X_1$. I = ϕ.
4.2) $j = 4$
4.3) $X_1 \geq X_4$. X_1 is not a minimal vector, so it is not a lower boundary point for (3,3; 2450). I = {1}.
4.1) $i = 2$
4.2) $j = 3$
4.3) $X_2 \ngeq X_3$ and $X_3 \ngtr X_2$. I = {1}.
⋮

The candidates X_1, X_2, and X_3 are deleted since $X_1 \geq X_4$, $X_2 \geq X_5$, and $X_3 \geq X_6$ after comparisons. Thus, $\Psi_{max} = \{X_4, X_5, X_6, X_7\}$ is the set of all lower boundary points for (3,3; 2450). To compute system reliability $R_{3,3;2450}$, let $S_1 = \{X|X \geq X_4\}$, $S_2 = \{X|X \geq X_5\}$, $S_3 = \{X|X \geq X_6\}$ and $S_4 = \{X|X \geq X_7\}$. Hence, $R_{3,3;2450} = \Pr\{S_1 \cup S_2 \cup S_3 \cup S_4\} = 0.676618532$ can be calculated by inclusion-exclusion principal. Note that $\Pr\{Y \geq X\} = \Pr\{y_1 \geq x_1\} \times \Pr\{y_2 \geq x_2\} \times \ldots \times \Pr\{y_n \geq x_{n+q}\}$ if $Y = (y_1, y_2, \ldots, y_{n+q})$. In particular, inclusion-exclusion principal states that $\Pr\{S_1 \cup S_2 \cup \ldots \cup S_r\} = \sum_i \Pr\{S_i\} - (-1)^2 \sum_{i<j} \Pr\{S_i \cap S_j\} - (-1)^3 \sum_{i<j<k} \Pr\{S_i \cap S_j \cap S_k\} - \ldots - (-1)^r \Pr\{S_1 \cap S_2 \cap \ldots \cap S_r\}$.The probability that the system processes (3 lots, 3 lots) under the budget \$2450 is 0.676618532.

6. COMPLEXITY ANALYSIS

The number of feasible solutions of $\sum_{j=1}^{m} f_j^1 = d_1$ and equation (8) are $\binom{m+d_1-1}{m}$ and $\binom{m+d_1-1}{m}\binom{m+d_2-1}{m}$, respectively. The number of solutions to constraints (8) – (10) is bounded by $\binom{m+d_1-1}{m}\binom{m+d_2-1}{m}$. Similarly, the number of X transformed according to equation (11) is bounded by $\binom{m+d_1-1}{m}\binom{m+d_2-1}{m}$. Hence, the algorithm needs $O((n+q)\binom{m+d_1-1}{m}\binom{m+d_2-1}{m})$ storage space in the worst case.

Each solution of equation (8) needs $O(m)$ time to test whether it satisfies $\left\lceil w_i^1 \sum_{a_i \in P_j} f_j^1 + w_i^2 \cdot \sum_{a_i \in P_j} f_j^2 \right\rceil \le M_i$ for each i and $O(m(n+q))$ time for all i. Hence, it takes $O(m(n+q)\binom{m+d_1-1}{m}\binom{m+d_2-1}{m})$ time to obtain all solutions to constraints (8) - (10) in the worst case. Since the number $\left\lceil w_i^1 \sum_{a_i \in P_j} f_j^1 + w_i^2 \sum_{a_i \in P_j} f_j^2 \right\rceil$ has been processed in step 2, it does not need any time to transform (F^1,F^2) into X via equation (11). In the worst case, the number of elements of Ω is $\binom{m+d_1-1}{m}\binom{m+d_2-1}{m}$, and so it takes $O((n+q)\binom{m+d_1-1}{m}\binom{m+d_2-1}{m})$ time to test an element of Ω whether it is minimal in Ω and $O((n+q)\binom{m+d_1-1}{m}^2\binom{m+d_2-1}{m}^2)$ time for all elements. Hence, the computational time complexity of the algorithm in the worst case is $O((n+q)\binom{m+d_1-1}{m}^2\binom{m+d_2-1}{m}^2) = O(m(n+q)\binom{m+d_1-1}{m}\binom{m+d_2-1}{m}) + O((n+q)\binom{m+d_1-1}{m}^2\binom{m+d_2-1}{m}^2)$. Note that m is less than $\binom{m+d_1-1}{m}\binom{m+d_2-1}{m}$. The computational complexity of the proposed algorithm is reflected by the number of MPs, number of nodes and edges, and the demand.

7. DISCUSSION

The system reliability can be treated as a performance index to measure the capability or quality level for a supply-demand system. Based on the properties of MP, an efficient algorithm to generate all lower boundary points for $(d_1,d_2;B)$ is proposed. System reliability can subsequently be calculated in terms of lower boundary points for $(d_1,d_2;B)$ by applying inclusion-exclusion principal. We can extend the proposed resolution procedure to multicommodity (more than two types of commodity) case easily. In our model the transmission cost c_i^k is not assumed to be linear in w_i^k. For the case that the transmission cost is only charged in terms of consumed capacity, c_i^k is linear in w_i^k. However, this condition is a special case of the proposed model.

The method discussed in Lin (2001[b]) studied the system reliability problem for single-commodity and unreliable nodes case but without any cost attribute. If we let w_i^k be constant for i, $d = w_i^1 d_1 + w_i^2 d_2$, B be unlimited, and treat the original problem (in TSFN) as a single-commodity case, then Lin's approach may be applied to evaluate $R_{d_1,d_2;B}$ (i.e., $R_{d,0;\infty}$). However, the following illustration indicates that the TSFN model cannot be simplified to a single-commodity model. We use the network of Figure 2 to illustrate the difference between them. If $w_i^1 = 1, w_i^2 = 3$ and $(d_1, d_2) = (1, 1)$, then $d = 4$. The capacity vector $X = (2, 2, 2, 0, 2, 0, 2, 2, 2, 2, 2, 2, 4, 4)$ permits flow $d = 4$ pass through P_1 and P_5. But it is obvious that the same capacity vector X cannot permit $(d_1, d_2) = (1, 1)$ since second type of commodity can pass through neither P_1 nor P_5.

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BIOGRAPHICAL SKETCH

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