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ROBUSTNESS OF DISPERSION CONTROL CHARTS IN SKEWED DISTRIBUTIONS

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This study examines the relative efficiency and the finite sample breakdown point of eight different estimators in Phase I of the control charting process when outliers occur in non-normal data. The performance of control charts based on these estimators is investigated by using average run lengths under four disturbances in three skewed distributions. The simulation result shows that control charts based on the modified biweight A estimator (D7) and the median of the absolute deviations (MAD) from the median are more robust than those in highly skewed distributions. In practice, in addition to robustness, computational simplicity is another important factor for practitioners when they are choosing control charts. It is thus suggested the control chart based on the MAD should be considered first due to its simplicity and robustness.

Keywords: breakdown point; non-normal distributions; outlier; relative efficiency; robustness

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1. INTRODUCTION

Control charts are constructed generally based on a normality assumption. However, Hogg (1974) argued that a few theoretical statisticians question the normality principle, and many applied statisticians have had qualms about the influence of basic assumptions. Janacek and Miekle (1997) also expressed that it is difficult to discover practical situations that the assumption is easily fulfilled, and non-normality situations are more observed in practice (Kao, 2012a; 2012b).

Outliers are an important issue in statistical inference. Many studies have discussed the causes of outliers, such as mistakes in the data input, unusual values due to sporadic assignable causes, or atypical observations that subsequently result in a small proportion of the original distribution being contaminated by other distributions (Abu-Shawiesh and Abdullah, 1999, Beckman and Cook, 1983; Hampel *et al*., 1986; Rocke, 1992). Hence, Outliers need to be specially borne in mind since a small departure from the assumed distribution can give rise to serious negative impacts on the efficiency of classical estimators. Outliers are also tricky to practitioners engaged in monitoring a process as they will both increase the variability and decrease the monitoring effectiveness of control charts (Riaz, 2008; Rocke, 1992).

Many control charts have used different estimators to strengthen their robustness in detecting outliers, including trimmed mean and standard deviation (Langenberg and Iglewicz, 1986), inter-quartile range (IR) (Rocke, 1989, 1992), Hodges-Lehmann and Shamos-Bickel-Lehmann's estimators (Abu-Shawiesh and Abdullah, 1999), Downton's estimators (Abu-Shawiesh and Abdullah, 2000), median absolute deviation (Abu-Shawiesh, 2009; Wu *et al*., 2002), Q estimators (Riaz, 2008), biweight A estimators (Tatum, 1997; Lax, 1985), M-estimate (Shahriari, Maddahi, and Shokouhi, 2009), screening method (Zwetsloot *et al*., 2015; Schoonhoven and Does, 2012), generalized likelihood ratio (GLR) test statistics (Zhang *et al*., 2017), randomness test (Shper and Adler, 2017), and coefficient of variation (Dawod *et al*., 2018).

Rocke (1989, 1992) showed that the effectiveness of the range chart is not good in detecting outliers and argued that an IR chart is more robust. Langenberg and Iglewicz (1986) proposed a trimmed range chart based on a trimmed range, which used a 25% trimmed mean of the subgroup ranges. Riaz (2008) proposed a Q chart that is formed by releasing the integer restriction of IR (Rocke, 1989, 1992). Wu *et al*. (2002) constructed Shewhart $\bar{X}$ control charts in accordance with seven estimators, including three proposed absolute deviations to the median.

Lax (1985) indicated that the biweight A estimator is most robust compared to absolute deviations from the median and trimmed standard deviation. Tatum (1997) first removed the median value of each subsample from that subsample, yielded a median-centered subsample, and applied the biweight A estimator to the pooled residual. Compared to other control charts based on various estimators, control charts using the modified biweight A estimator show superior robustness in four types of disturbances. Zwetsloot *et al*. (2015) and Schoonhoven and Does (2012) pointed out that control charts based on estimators with screening estimators are more robust than various robust dispersion estimators. Shahriari, Maddahi, and Shokouhi (2009) showed that a standard deviation control chart using M-estimate has better robustness than other estimators. All of the aforementioned studies analyze the efficiency of the estimators or the robustness of control charts in the normality assumption. Zhang *et al*. (2017) argued that a chart with the GLR statistics is able to detect the decrease in variability effectively. Shper and Adler (2017) showed that the order of points or data randomness could seriously impact the performance of phase I control charts performance. Therefore, the test (average of the moving range to standard deviation) is proposed to evaluate the randomness. Dawod *et al.* (2018) pointed out estimators based on the coefficient of variation (CV) have good efficiencies when the process mean or standard deviation is not constant.

A few studies discussed the robustness of control charts in non-normality. Abu-Shawiesh and Abdullah (1999, 2000) and Abu-Shawiesh (2009) investigated the shift detection effectiveness of control charts based on Hodges-Lehmann and Shamos-Bickle-Lehmann, Downton's estimator, and sample median and median absolute deviation, respectively in four distributions–uniform, normal, double exponential, and Cauthy, Nazir, Riaz, and Does (2014) discussed properties of the design structure of cumulative sum (CUSUM) control charts based on existing estimators in students' *t*, logistic, and gamma distributions. Human *et al.* (2011) investigated the in-control robustness of an exponentially weighted moving average (EWMA) in accordance with the sample standard deviation and the moving range in these distributions–student's *t*, uniform, right triangular, gamma, symmetric bi-modal, and asymmetric bi-modal as well as the contaminated normal. Maravelakis, Panaretos, and Psarakis (2005) tested the detection effectiveness of EWMA charts based on five schemes in students' *t*, gamma, and normal distributions.

The majority of the aforementioned studies investigate the influence of symmetric non-normal or contaminated normal distributions on control charts, and a gamma distribution is only considered to be asymmetric non-normal. Tatum (1997) thought that four types of disturbances could represent outlier statuses, diffuse symmetric, diffuse asymmetric variance, localized variance, and diffuse mean disturbances. However, all of the studies check the robustness of estimators in process shifts in the non-normal processes, not considering Tatum's (1997) disturbances.

Because of the lack of complete disturbance investigation in existing estimators, the aim of this study is to examine the efficiency of eight existing estimators under four disturbances in skewed distributions and to study the robustness performance of dispersion control charts using these estimators. Developing new control charts with new estimators is not the focus of this study. The remainder of this paper is organized as follows. Section 2 describes the estimators of process dispersion. Section 3 elaborates the evaluation of estimators in different circumstances. Section 4 analyzes the performance of Phase II control charts, and the last section concludes the findings of this study.

2. DESCRIPTION OF ESTIMATORS OF PROCESS DISPERSION

Eight estimators will be investigated in skewed distributions. Assume that the X_{ij} $(i = 1, \cdots, m; j = 1, \cdots, n)$ is the Phase I data with the subgroup size *n* and the number of subgroups *m*.

1) Modified biweight A estimator with *c*=7 (D7)

The modified biweight A estimator, which is robust to estimate the process standard deviation, was proposed by Tatum (1997). Define the residual as $Y_{ij} = X_{ij} - \tilde{X}_i$. Here $\tilde{X}_i$ is the sample median. If n is odd, subtracting out the median will result in one zero value, which is dropped. When n is even, the total number of median-centered subsample values is $\acute{m} = mn$, and $\acute{m} = (n-1)m$ when n is odd. Tatum's (1997) estimator can be written as:

$$S_c^* = \frac{n\acute{m}}{(\acute{m}-1)^{1/2}} \frac{\left(\sum_{i=1}^{m}\sum_{j:|u_{ij}|<1} Y_{ij}^2 (1-{u_{ij}}^2)^4\right)^{1/2}}{\left|\sum_{i=1}^{m}\sum_{j:|u_{ij}|<1} (1-{u_{ij}}^2)(1-5{u_{ij}}^2)\right|}, \tag{1}$$

where c is a tuning constant, $u_{ij} = h_i Y_{ij} / (c\widetilde{M})$ and $\widetilde{M}$ is the median of all residuals.

Here $h_i = \begin{cases} 1, E_i < 4.5 \\ E_i - 3.5,\ 4.5 < E_i \le 7.5, \\ c, E_i > 7.5 \end{cases}$ and $E_i = IR_i / \widetilde{M}$. IR_i, is the inter-quartile range of i^{th} subgroup. An unbiased estimator of σ can be shown as $\hat{\sigma}^{I,1} = S_c^*/d^*$.

2) Inter-quartile range (IR)

Let the $X_{i(j)}$ be the j^{th} order observation in the subgroup i. According to Rocke (1989), the IR of the i^{th} subgroup is $IR_i = X_{i(u)} - X_{i(l)}$, where $l = [n/4] + 1$, and $u = n - l + 1$. Here $[\cdot]$ is the floor function. That is, $[\cdot]$, is the greatest integer that is smaller than or equal to x.

The mean of the sample interquartile ranges is expressed as

$$\overline{IR} = \sum_{i=1}^{m} IR_i/m \tag{2}$$

Hence, the unbiased estimator ($\hat{\sigma}^{I,2}$) of σ is equal to $\overline{IR}/d_{2IR}$.

3) 25% trimmed average of the IR (IR25)

Rocke (1989) also considered the other IR version, the 25% trimmed average of the IR. The version is denoted as IR25. The mean-sample 25% trimmed average of the IR is given by.

$$\overline{IR}_{25} = \textstyle\sum_{i=k+1}^{m-k} IR_i/[m/2]. \tag{3}$$

Therefore, IR25 can be written as $\hat{\sigma}^{I,3} = \overline{IR}_{25}/d_{2IR25}$, where $k = [0.25m]$ and $[\cdot]$ is the floor function.

4) Median of the absolute deviations from the median (MAD)

Hampel (1974) proposed a robust estimator based on the median of the absolute deviations from the median. The estimator is defined as $\text{MAD}_i = Median|X_{ij} - \tilde{X}_i|$. The mean MAD is given by

$$\overline{MAD} = \sum_{i=1}^{m} MAD_i/m. \tag{4}$$

Therefore, the unbiased estimator ($\hat{\sigma}^{I,4}$) of σ is equal to $\overline{MAD}/d_{2MAD}$.

5) Gini's estimator (Gini)

The estimator was proposed by Gini and is the same as the Downton estimator (Downton, 1966) and the probability-weighted moment's estimator (Muhammad *et al.*, 1993). The estimator is written as $G_i = \sum_{j=1}^{n-1}\sum_{l=j+1}^{n} |X_{ij} - X_{il}|/(n\,(n-1)/2)$. The unbiased estimator of σ can be shown as $\hat{\sigma}^{I,5} = \bar{G}/d_{2Gini}$.

Here $\bar{G} = \sum_{i=1}^{m} G_i/m.$ (5)

6) Range (R)

Define the sample range as $R_i = X_{i(n)} - X_{i(1)}$ and the mean-sample range is

$$\bar{R} = \textstyle\sum_{i=1}^{m} R_i/m. \tag{6}$$

The unbiased estimator of σ can be expressed as $\hat{\sigma}^{I,6} = \bar{R}/d_2$.

7) Sample standard deviation (S)

The estimator based on the mean-sample standard deviation can be shown as:

$$\bar{S} = \sum_{i=1}^{m} S_i/m, \tag{7}$$

where $S_i = \left[\sum_{j=1}^{n}(X_{ij} - \bar{X}_i)^2/(n-1)\right]^{\frac{1}{2}}$ and $\bar{X}_i$ is the average of the *i*th subgroup. The unbiased estimator of σ can be expressed as $\hat{\sigma}^{I,7} = \bar{S}/c_4$.

8) Standard deviation of trimmed mean (TR)

Define the ζ as the trimmed rates and $\zeta \in [0, 1/2)$. The variance of the trimmed mean is expressed as:

$$TRV_i = \frac{1}{n-2k}\left(\sum_{j=k+1}^{n-k}(X_{(j)} - \bar{X}_\tau)^2 + k(X_{(k)} - \bar{X}_\tau)^2 + k(X_{(n-k+1)} - \bar{X}_\tau)^2\right),$$

where $k = [\zeta(n-1)]$ and $[X]$ is the floor function. Here the ζ is fixed as 0.25.
Here $\bar{X}_\zeta = \sum_{j=k+1}^{n-k} X_{(j)}/(n-2k)$. Therefore, the standard deviation of trimmed mean is TRS_i=$\sqrt{TRV_i}$. The unbiased estimator of σ can be expressed as $\hat{\sigma}^{I,8} = \overline{TRS}/d_{2TR}$, where

$$\overline{TRS} = \sum_{i=1}^{m} TRS_i/m. \tag{8}$$

Tatum (1997) expressed that when no disturbances exist, the estimator with c = 7 can lose some efficiency but can increase efficiency when disturbances are present. Hence, the c is given as 7 in this study.

The values, d^*, d_{2IR}, d_{2IR25}, d_{2MAD}, d_{2Gini}, d_2, c_4, and d_{2TR}, are calculated from 10^6 simulations for each grade of skewness (α_3) by Mathematica 7.0 software when the sample sizes are considered as 5 and 10.

3. EVALUATION OF ESTIMATORS

The efficiency of eight estimators will be investigated in four types of disturbances and three types of distributions, beta (BT), inverse Gaussian (IG), and Weibull (WB). The BT is negatively skewed, and the remainders are positively-skewed.

Tatum (1997) pointed out that any observation that is equally likely upset is called a diffuse disturbance; all members of a particular subsample or subsamples are perturbed is a localized disturbance. This study considers the data scenarios by following Tatum's four types of disturbances. These disturbances are depicted as follows (Tatum, 1997).

3.1 Four Types of Disturbances

1) A model for diffuse symmetric disturbances in which each observation has a 95% probability of being generated from the $\theta(\alpha, \beta)$ distribution with mean (μ) and standard deviation (σ) and a remaining 5% probability of being generated from $\theta(\tau\ \alpha, \omega\ \beta)$ distribution with μ and rσ. The $\theta(\alpha, \beta)$ is the beta, inverse Gaussian, or Weibull distributions. Here $\mathrm{r} = 1, 1.25, \cdots, 2$ for beta distribution and $\mathrm{r} = 1, 2, \cdots, 5$ for the remainders. Since the beta random variates are limited between 0 and 1, the BT's standard deviation cannot arbitrarily shift like the three positively skewed distributions; and so the r scales of both skewed kinds are different. For example, when $\alpha_3 = -1$, the $\mu = 0.787$ and $\sigma = 0.171$ for BT(3.7, 1). When the mean of contaminated data keeps the same values as μ, it is impossible to change drastically for rσ. The r-value cannot be larger than 2.92 because the maximum of contaminated σ is 0.5 if β is fixed as 1.
2) A model for diffuse asymmetric variance disturbances in which each observation is generated from the $\theta(\alpha, \beta)$ distribution with μ and σ and has a 5% probability of having a multiple of a gamma ($\tau\ \alpha, \omega\ \beta$) variable with μ and $\gamma\ \sigma$ added to it, with $\mathrm{r} = 1, 1.25, \cdots, 2$ for beta distribution and $\mathrm{r} = 1, 2, \cdots, 5$ for the remainders.
3) A model for localized variance disturbances in which all observations in two subgroups are generated from the $\theta(\tau\ \alpha, \omega\ \beta)$ distribution with μ and rσ, and the remaining subgroups contain observations drawn from the $\theta(\alpha, \beta)$ distribution with μ and σ. Here $\mathrm{r} = 1, 1.25, \cdots, 2$ for beta distribution and $\mathrm{r} = 1, 2, \cdots, 5$ for the remainders.
4) A model for diffuse mean disturbances in which each observation has a 95% probability of being generated from the $\theta(\alpha, \beta)$ distribution with μ and the σ and a remaining 5% probability of being generated from the $\theta(\tau\ \alpha, \omega\ \beta)$ distribution with $\delta\ \mu$ and σ. Here $\delta = 0.25, 1.25, \cdots, 4.25$ for beta distribution and $\delta = 0.5, 1.5, \cdots, 4.5$

for the remainders. The δ scales of BT are as different as those of the positively skewed distributions. The consideration of setting a different δ scale is similar to that of setting a different r scale.

5) In Models1-3, the mean (μ_c) of $\theta(\tau\alpha, \omega\beta)$ and of gamma $(\tau\alpha, \omega\beta)$ is the same as the mean of $\theta(\alpha, \beta)$, but their standard deviation (σ_c) is the r times of σ. For Model4, two kinds of distributions have the same σ, but $\mu_c = \delta\mu$. To generate the contaminating distributions of Models 1-4, the scale and shape parameters of a skewed distribution have to be reset after calculating the value of τ and ω. Given the α, β, μ, and σ, the τ and ω can be determined by $\sigma_c = r\sigma$ or $\mu_c = \mu$.

3.2 Relative Efficiency of Estimators

Park and Wang (2020) point out that relative efficiency (RE) is one kind of good measurement for evaluating the performance of unbiased estimators. RE is defined as

$$\text{RE}\left((\hat{\sigma}^{I,i} \middle| \hat{\sigma}^{I,1})\right) = \frac{V(\hat{\sigma}^{I,1})}{V(\hat{\sigma}^{I,i})}, \quad (9)$$

where $\hat{\sigma}^{I,1}$ is a reference estimator and is fixed as D7. For $i = 2, \cdots, 8$, $\hat{\sigma}^{I,i}$, are the other estimators in Section2. RE is evaluated by using 10^6 simulation runs.

Figures 1-3 show the values (%) of RE (REs) of the eight estimators for BT, IG, and WB when *n*=5. They clearly show REs of D7, IR, IR25, and MAD are close regardless of the distributions and the disturbances, whereas the REs of Gini, R, S, and TR are near. The study defines the prior four estimators as the first type of index; the remainders are the second type of index. From the results presented in these Figures, a number of observations are made:

1) Diffuse symmetric disturbances

i. The REs of the first type of estimators are lower than those of the second type when $r \leq 2$. On the contrary, the first type significantly has higher REs than the second type when $r > 2$. The result testifies that the second type is easy to be affected by diffuse symmetric disturbances. The first type adopts the trim-type method, so these estimators are more efficient.

ii. In the negatively skewed distribution, Gini performs best; IR25 is the worst. It is worth noting that the second type performs better than the first type in the small values of r. The second type is merely suitable for small disturbances.

iii. In positively skewed distributions, D7 has the best efficiency for $\alpha_3 = 1$ but is not best for $\alpha_3 = 2.5$, particularly for in WB. IR and MAD perform best in IG and WB for $\alpha_3 = 2.5$. The performance of TR is the worst among all of these estimators.

2) Asymmetric variance disturbances

i. When data is contaminated by the gamma distribution, the second type of index has superior efficiency in the disturbance compared to the first type of index. Because the mean of the gamma distribution is fixed as 0.64, the gamma $(0.64\tau, \omega\beta)$ is unlikely to generate very large data. It represents that the original distribution can be impacted slightly by this gamma distribution. For the first type of index, the estimators trim some smaller and larger data which could not be contaminated so could obtain less efficiency. The result points out the robust estimators that have been validated are not always robust. Their efficiency depends on contaminations that are generated by which type of distribution.

ii. For all of the estimators, the performance of Gini is the best. Compared to the four first-type estimators, D7 has the best efficiency in BT but is almost the worst for $\alpha_3 = 2.5$ of WB. IR25 is almost the worst regardless of the distributions and disturbances.

3) Localized variance disturbances

i. In BT, Gini is the best; IR25 is the worst.

ii. MAD is better in IG, particularly for $\alpha_3 = 2.5$. D7 is worst in the first type of index when $\alpha_3 = 2.5$, but it is best in WB when $\alpha_3 = 1$. Because the highly skewed IG and WB are easier to generate extreme values, particularly for WB, the weight (h_j) of the contaminated subgroup will significantly increase. The result will result in the big values of D7 and increase its variance. TR performs worst in all of the positively skewed distributions.

4) Diffuse mean disturbances

i. The second type of index is better than the first when $\delta \leq 2.25$ in BT or when $\delta \leq 1.5$ in IG and WB, when δ>2.25 or $\delta > 1.5$, the REs of the second type declined sharply.

ii. D7 is best in the first type of index in BT. MAD performs best for $\alpha_3 = 1$, followed by IR25 in IG; D7 is best in IG for $\alpha_3 = 2.5$. In WB, D7 has the best efficiency for $\alpha_3 = 1$; D7 performs better for $\delta \geq 2.25$ when $\alpha_3 = 2.5$.

iii. Compared to all of the first-type estimators, D7 has the best efficiency for $\alpha_3 = 1$ but is not best for $\alpha_3 = 2.5$. MAD has better efficiency than D7 for $\alpha_3 = 2.5$. TR still obtains the worst performance for all of the positively skewed distributions.

3.3 Finite Sample Breakdown Point (FBP)

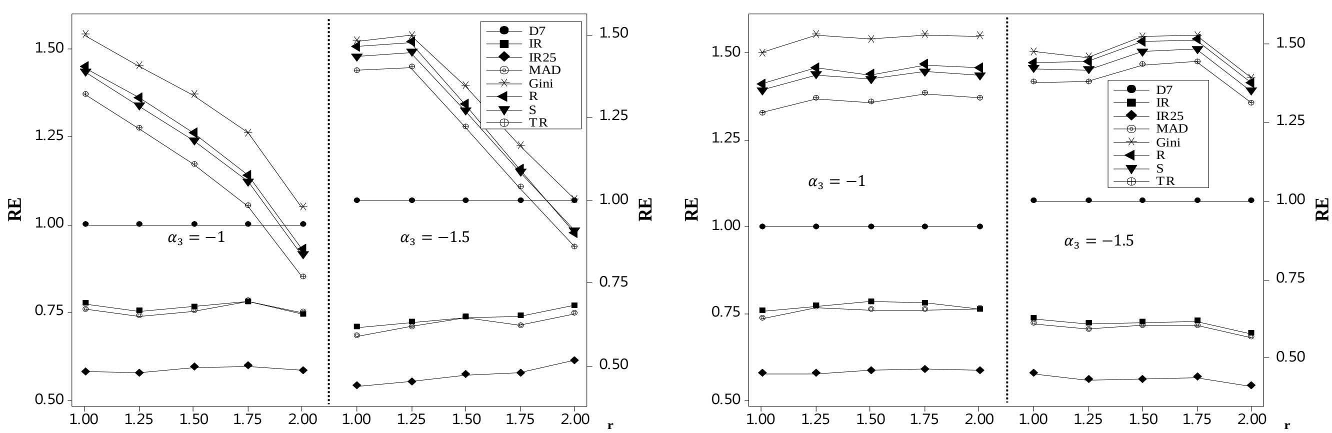
Hampel (1968) proposed the breakdown point and gave it an asymptotic definition. The term is the smallest amount of contamination that may make an estimator take on arbitrarily aberrant values (Huber and Ronchetti, 2009). To be of practical implementation, Donoho (1982), and Donoho and Huber (1983) coined a simplified version for finite samples.

The IR is defined as the difference between the second smallest and the second-largest observations for $4 \leq n \leq 7$; the IR is the difference between the third smallest and the third largest for $8 \leq n \leq 11$. When $n \leq 7$, IR can resist the manipulation of a single observation. However, the manipulation of a second observation in the same subgroup can draw IR beyond any bound. Therefore, the FBP of IR is $1/(mn)$, for $n \leq 7$. For $8 \leq n \leq 11$, its FBP is $2/(mn)$, because this estimator will not break down till a third observation in the same subsample is manipulated. Because TR uses the same 25% trimmed method as IR in the same subgroup, the FBP of TR is the same as that of IR.

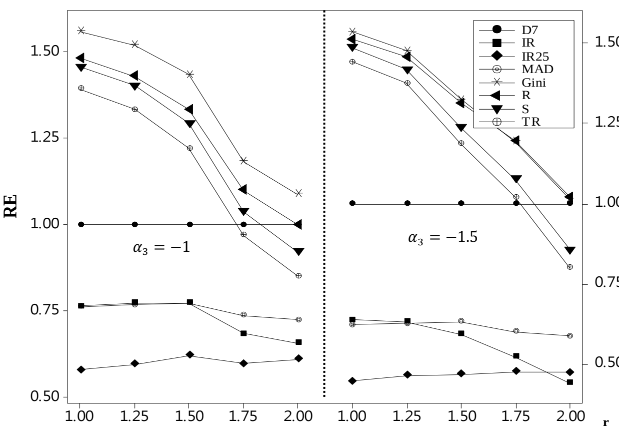
When m = 30, n = 5, the 7 largest and the 7 smallest IR values are trimmed off for IR25. The manipulation of two observations in each of the eight subgroups can break down IR25. When k = 0.25m is an integer and $n \leq 7$, the FBP of IR25 is $(2k-1)/(mn)$. When $8 \leq n \leq 11$, the FBP is $(3k-1)/(mn)$, because three observations are needed in k subgroups to break down the IR25.

Because the modified biweight A uses the MAD of the subsample-centered observations, $\widetilde{M}$. D7 will receive the FBP of the $\widetilde{M}$. The FBP of $\widetilde{M}$, depends on the function of m and n. According to Tatum (1997), the FBP is $1/4 - 1/(mn)$ when m and n are even; the point is $1/4 + 1/4n - 1/(mn)$, when m is even, and n is odd. The FBP of $\bar{G}$, $\bar{R}$, and $\bar{S}$ are 0 since the use of just one data can be manipulated to drive these estimators beyond any given bound.

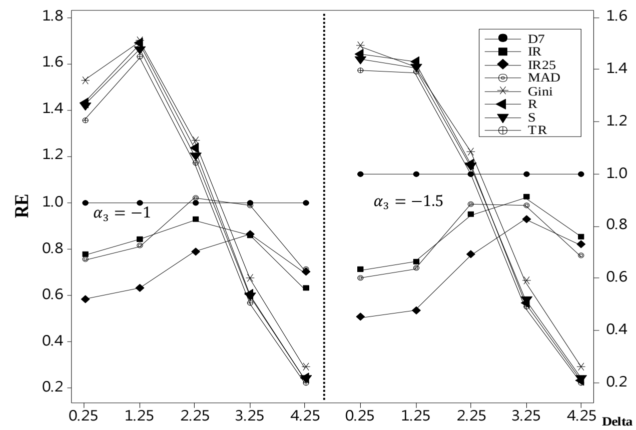
Table 1 lists the FBP for m = 30 for D7, IR, IR25, MAD, Gini, R, S, and TR. The FBP of D7 and MAD is best, followed by IR25. $\bar{G}$, $\bar{R}$, and $\bar{S}$ has the worst FBP.



(a) diffuse symmetric disturbances (b) asymmetric variance disturbances

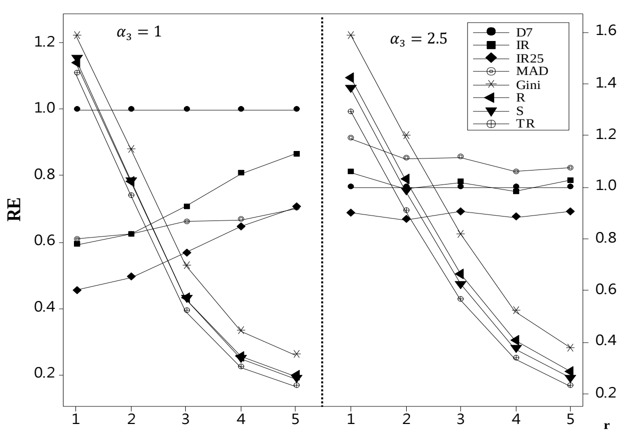


(c) localized variance disturbances

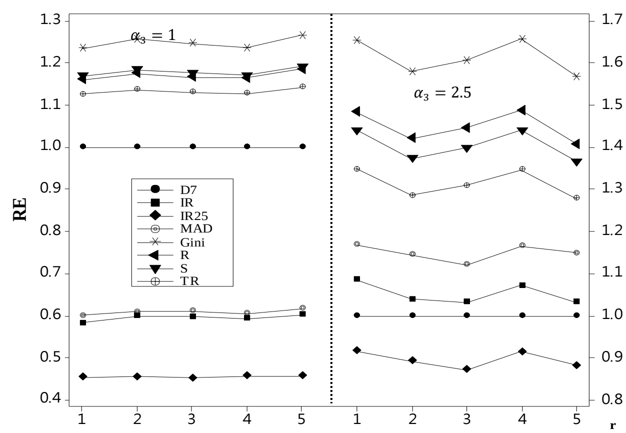


(d) diffuse mean disturbances

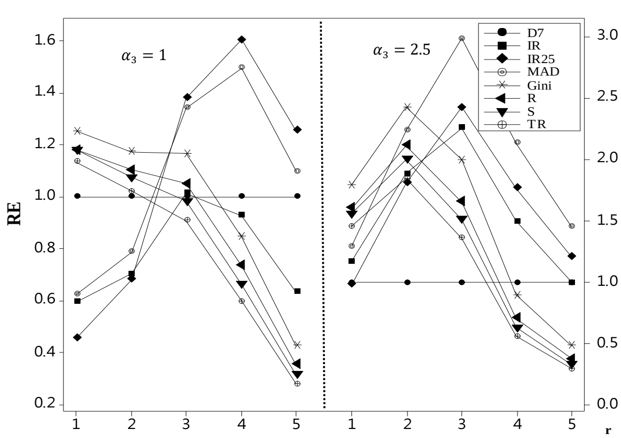
Figure 1. RE of estimators (%) in beta distribution for n=5



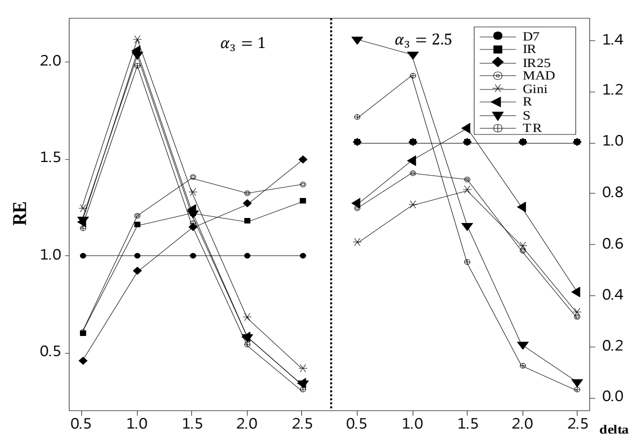
(a) diffuse symmetric disturbances



(b) asymmetric variance disturbances

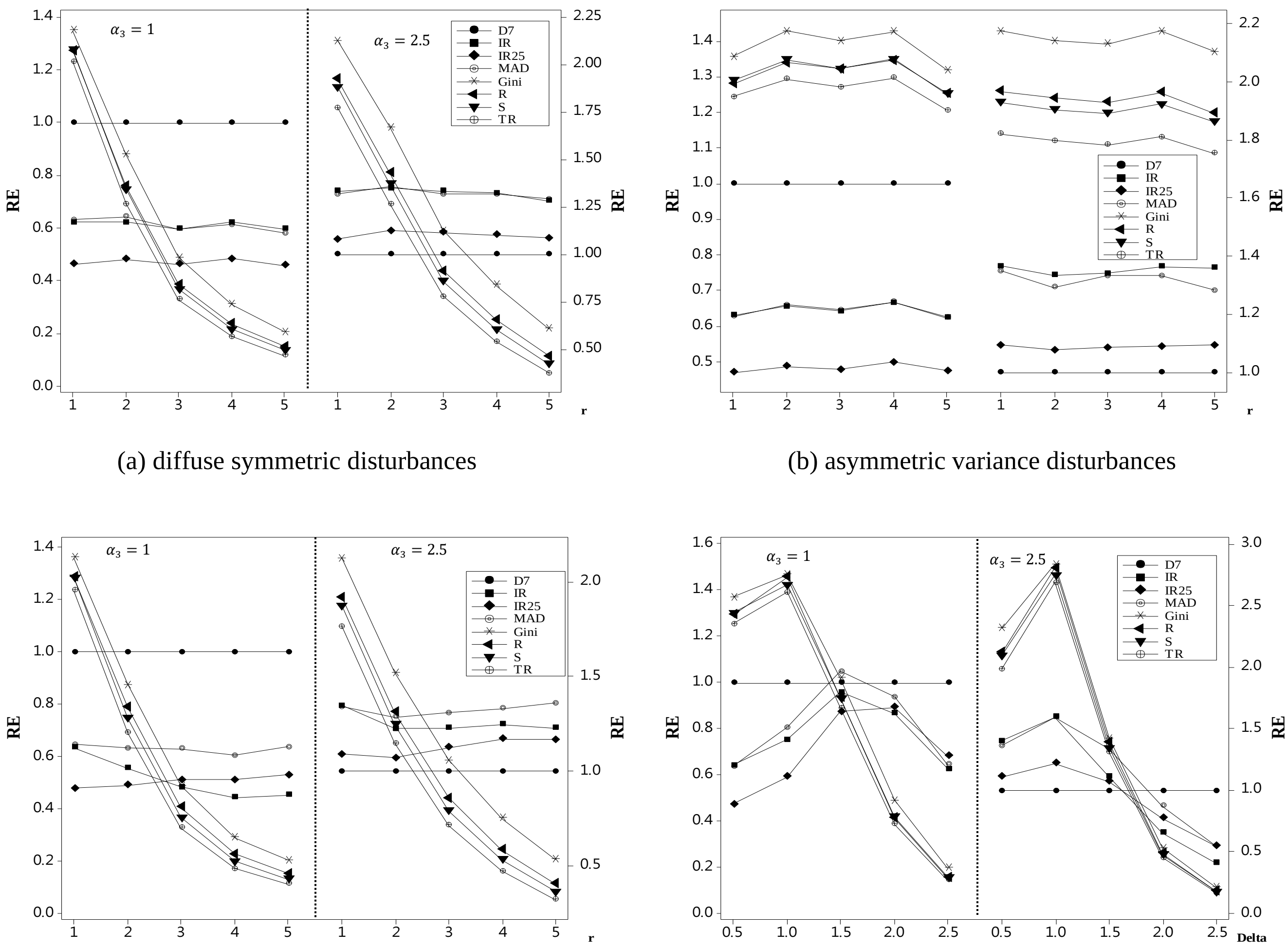


(c) localized variance disturbances



(d) diffuse mean disturbances

Figure 2. RE of estimators (%) in inverse Gaussian distribution for n=5



(a) diffuse symmetric disturbances (b) asymmetric variance disturbances

(c) localized variance disturbances (d) diffuse mean disturbances

Figure 3. RE of estimators (%) in Weibull distribution for n=5

Table 1. FBP of estimators (%) for m=30

n	D7	IR	IR25	MAD	Gini	R	S	TR
4	24.2	.8	10.8	24.2	.0	.0	.0	.8
5	29.3	.7	8.7	29.3	.0	.0	.0	.7
6	24.4	.6	7.2	24.4	.0	.0	.0	.6
7	28.1	.5	6.2	28.1	.0	.0	.0	.5
8	24.6	.8	8.3	24.6	.0	.0	.0	.8
9	27.4	.7	7.4	27.4	.0	.0	.0	.7
10	24.7	.7	6.7	24.7	.0	.0	.0	.7

4. PERFORMANCE OF PHASE II CONTROL CHARTS

The effect of these estimators on the Phase II performance of standard deviation control charts will be evaluated in three kinds of skewed distributions. The same Phase I estimators ($\hat{\sigma}^I$) are considered in constructing control charts, and the multipliers, r, and δ are fixed as 4 to simulate the contaminated cases.

4.1 Simulation Procedure

The study considers that the process standard deviation has different shifts when the process is out-of-control. The shift grade of process standard deviation is set as λσ, with λ= 0.6, 1, 1.2, and 1.4. Meanwhile, the robustness of control charts is examined in the four grades of skewness, $\alpha_3 = -1.5, -1, 1,$ and 2.5.

Let $\widehat{LCL} = L\hat{\sigma}^I$ and $\widehat{UCL} = U\hat{\sigma}^I$ be the lower control limits and upper control limits, respectively. L and U are the constants of the control chart. Meanwhile, define $\hat{\sigma}_i^{II}$ as the value of the Phase II estimator on the i^{th} subgroup and let $\hat{\sigma}_i^{II}$ estimate λσ. Denote that H_i is the event that $\hat{\sigma}_i^{II}$ falls outside the control limits. Therefore, the probability of H_i can be expressed as $P(H_i|\hat{\sigma}^I) = P\left(\hat{\sigma}_i^{II} < \widehat{LCL}\ or\ \hat{\sigma}_i^{II} > \widehat{UCL}\ \middle|\hat{\sigma}^I\right)$.

If the distribution of the run length (RL) is geometric with parameter $P(H_i|\hat{\sigma}^I)$, the conditional ARL is written as $\mathrm{E}(\mathrm{RL}|\hat{\sigma}^I) = \frac{1}{P(H_i|\hat{\sigma}^I)}$.

Compared with the conditional RL distribution, the unconditional RL distribution considers the random variability introduced into the charting through parameter estimation. The unconditional RL distribution can be acquired by averaging the conditional RL distribution of the reference sample (X_{ij}) over all possible parameter estimates. Hence, the unconditional ARL and standard deviation of the run length (SDRL) can be expressed as (Schoonhoven and Does, 2012)

$$\mathrm{ARL} = \mathrm{E}\frac{1}{P(H_i|\hat{\sigma}^I)} \tag{10}$$

and

$$\mathrm{SDRL} = \sqrt{2\mathrm{E}\left(\frac{1}{p(H_i|\hat{\sigma}^I)}\right)^2 - \left(\mathrm{E}\left(\frac{1}{p(H_i|\hat{\sigma}^I)}\right)\right)^2 - \mathrm{E}\left(\frac{1}{p(H_i|\hat{\sigma}^I)}\right)}. \tag{11}$$

In order to acquire $P(H_i|\hat{\sigma}^I)$, $\mathrm{E}(\mathrm{RL}|\hat{\sigma}^I)$ and sufficiently small estimated standard errors for ARL, the simulation runs are fixed as 10^6. The unconditional values of ARL and SDRL can be obtained by averaging these values.

4.2 Simulation Results

ARL and SDRL are determined in the in-control scenario ($\lambda = 1$) and in the out-of-control scenario ($\lambda \neq 1$). Tables 2-6 list the ARL and SDRL values of BT, IG, and WB. Table 2 shows that ARL is close to their expected values of 370 when the process is in control. Tables 3-6 summarize the ARL of control charts in the four types of disturbances in the three distributions. From the results presented in these tables, a number of observations are made:

1) Diffuse symmetric disturbances

i. Regardless of sample sizes, skewed magnitudes, and the types of distributions, the control charts based on the first type of index are superior in ARL compared to those based on the second type of index. For the negatively skewed distribution, control charts based on MAD and D7 perform best. For the positively skewed distributions, the control chart based on D7 has the best performance, followed by IR or IR25. Compared to the studies of Tatum (1997) and Schoonhoven & Does (2012), the performance of D7 in skewed distributions is as well as in the normality assumption. Control charts based on Gini and R have the worst robustness. Barnett *et al.* (1967) argued that the Downton estimator owns high efficiency and robustness against outliers in a normal distribution. However, in this study, the ARL of control charts based on Gini is as easily affected by disturbances as those based on R and S.

ii. Comparing $\alpha_3 = -1$ and $\alpha_3 = -1.5$, the out-of-control ARL (ARL_1) of all of the estimators in $\alpha_3 = -1.5$ is closer to the intended in-of-control ARL (ARL_0) than that in $\alpha_3 = -1$. The ARL_1 to ARL_0 in $\alpha_3 = 2$ is also nearer compared to that in $\alpha_3 = 1$. That is, they are easy to generate remarkably larger or smaller data in the highly skewed distributions. When the significant extreme values are contaminated, the changes of the values cannot be too drastic. In contrast to the highly skewed distributions, their values have a great change in the contaminated lowly-skewed distributions. Hence, the performances of control chars can be more affected by the disturbances in contaminated lowly skewed distributions.

iii. Besides on WB, the value of ARL (ARLs) of the first type of index are larger than those of the second type of index when $\lambda = 0.6$. It is very clear that the ARLs of the control charts are significantly larger in WB than those in IG when $\alpha_3 = 2$ and $\lambda = 0.6$. For example, for $\lambda = 0.6$ in Table 3, ARL ≥ 557 when *n*=10; ARL ≥ 830 when *n*=5. Given α_3 the contaminated $\mathrm{WB}(\tau\alpha, \omega\beta)$ has a larger standard deviation than the contaminated $\mathrm{IG}(\tau\alpha, \omega\beta)$. It represents that the contaminated $\mathrm{WB}(\tau\alpha, \omega\beta)$ will generate more extreme data than the others. Therefore, the tolerance of control limits will be broader after employing the contaminated data, so the control charts will result in larger ARLs in WB.

iv. Tables 3-6 show that the control chart based on MAD has the best robustness than the others in the highly skewed distributions and the large variances. However, the performances of the control charts based on the first type of index are not significantly different in the lowly-skewed distributions.

2) Asymmetric variance disturbances

Regardless of the kinds of estimators, their ARLs are close to their ARL_0. The control charts are slightly affected by the asymmetric variance disturbances, which are made from the gamma $(0.64\tau, \omega\beta)$ because the distribution is unlikely to generate extreme value.

3) Localized variance disturbances

i. For BT, the D7 chart performs best, followed by MAD. For the positively skewed distributions, the control charts based on IR or IR 25 have the best robustness. Nevertheless, the Gini, R, or S charts perform worst when a process presents localized variance disturbances.

ii. The ARL performances of the estimators in the localized disturbances are similar to those in diffuse symmetric disturbances for the positively skewed distributions. Compared to Table 2, ARLs of D7 significantly increase in the disturbances when $\lambda = 0.6$; ARLs of D7 significantly decrease when $\lambda \geq 1$. Regardless of sample sizes, the robustness of the control chart based on D7 is best among these control charts.

4) Diffuse mean disturbances

For the negatively skewed distribution, D7 and MAD chart performs best; for the positively skewed distributions, the robustness of the control chart based on D7 is best. But the Gini chart is as bad as R or S charts. Compared with Table 2, Table 5 shows that the disturbances decline ARLs of control chars when $\lambda = 0.6$; ARLs increase when $\lambda \geq 1$. Particularly, the phenomenon is very obvious in WB.

Table 2. ARL and SDRL of control charts when uncontaminated data in Phase I for *m*=30

		n=5								*n*=10							
		ARL				SDARL				ARL				SDARL			
	λ =	0.6	1	1.2	1.4	0.6	1	1.2	1.4	0.6	1	1.2	1.4	0.6	1	1.2	1.4
								$\alpha_3 = -1.5$									
BT	D7	696	369	103	33.6	693	406	126	35.6	670	369	66.6	19.9	688	392	70.8	20.2
	IR	565	370	112	41.5	578	410	141	48.6	280	369	88.0	29.6	300	386	101	31.8
	IR25	595	371	124	44.9	642	411	163	55.0	279	370	96.2	30.5	327	394	110	33.8
	MAD	681	370	117	39.9	716	401	137	43.1	429	370	88.4	27.7	486	391	94.4	26.5
	Gini	547	370	75.7	24.0	608	410	92.2	27.3	188	371	60.3	15.7	220	385	69.5	16.9
	R	568	371	79.5	24.8	635	390	98.3	27.4	209	370	76.2	21.2	257	383	87.9	23.5
	S	528	369	78.2	24.3	576	391	90.5	26.9	183	370	61.7	14.8	225	382	73.0	16.7
	TR	561	372	81.9	24.9	635	397	92.1	28.0	221	371	73.5	20.7	261	382	83.0	22.3
								$\alpha_3 = -1.0$									
BT	D7	551	369	90.5	22.5	557	410	111	24.2	420	370	42.9	10.0	416	406	50.3	9.9
	IR	457	370	90.5	24.7	477	412	133	33.2	166	370	63.2	14.6	173	408	83.3	16.0
	IR25	462	371	104	28.1	479	415	164	37.3	161	371	65.7	14.9	185	397	87.2	17.3
	MAD	555	369	96.7	26.4	564	417	121	28.8	234	371	64.6	15.9	266	397	71.4	16.0
	Gini	326	371	60.8	14.6	351	420	87.7	19.3	73.5	370	41.7	8.2	83.0	410	57.7	8.8
	R	345	369	69.7	16.4	360	425	119	27.1	80.1	369	68.9	14.4	90.7	419	129	36.2
	S	314	372	63.7	13.4	333	416	94.2	19.0	71.5	370	39.0	6.9	79.9	416	53.0	8.1
	TR	322	371	65.4	14.3	348	423	103	19.2	106	370	47.3	8.9	122	395	64.3	9.8
								$\alpha_3 = 1.0$									
IG	D7	435	370	117	43.2	445	403	143	53.7	325	368	75.6	23.6	331	400	95.6	26.7
	IR	415	370	137	58.8	428	408	179	76.4	137	371	108	34.8	153	390	133	43.3
	IR25	416	369	146	64.0	423	414	203	88.8	136	367	112	37.0	151	381	146	46.3
	MAD	420	369	148	67.6	438	410	196	87.9	144	369	117	41.1	158	381	148	51.0
	Gini	189	368	96.2	32.8	206	384	114	36.4	39.7	370	70.5	18.2	45.4	385	86.5	19.9
	R	182	369	109	35.5	203	382	132	41.2	40.6	371	99.5	28.2	45.9	383	112	31.4
	S	178	371	107	35.7	194	395	128	41.0	38.9	372	82.7	22.9	43.0	382	93.6	24.6
	TR	181	370	119	39.7	194	375	139	45.8	72.3	370	89.4	27.3	79.3	380	107	29.7
WB	D7	503	368	101	40.9	503	410	122	49.2	381	369	72.1	26.9	377	396	88.8	31.6
	IR	471	369	115	48.9	485	415	150	60.8	181	368	90.9	33.4	199	387	110	40.0
	IR25	470	370	125	50.3	479	422	165	65.3	173	370	98.8	36.2	209	392	148	65.4
	MAD	487	372	141	68.2	511	420	183	83.0	194	370	124	57.4	196	390	119	43.1
	Gini	263	371	63.6	20.2	281	388	76.4	22.0	66.1	368	44.1	12.1	73.2	387	52.3	12.7
	R	248	371	63.8	22.0	270	378	76.2	23.4	65.2	371	53.3	15.3	72.4	380	48.5	12.0
	S	258	369	65.8	20.9	273	390	75.1	22.2	62.4	369	43.9	12.1	75.3	380	56.9	15.9
	TR	259	370	64.0	21.4	281	387	74.1	22.4	103	369	63.6	20.4	115	381	71.5	21.5
								$\alpha_3 = 2$									
IG	D7	572	371	165	89.2	619	409	202	103	505	368	138	68.0	524	401	167	81.2
	IR	526	369	166	82.9	544	397	200	100	283	369	139	61.5	311	386	152	69.1
	IR25	549	370	175	91.3	593	324	217	112	275	368	146	67.5	315	385	169	77.5
	MAD	583	372	188	110	596	405	216	129	308	370	158	86.9	340	390	176	95.6
	Gini	380	368	120	49.5	403	387	136	54.0	142	368	94.8	33.3	179	381	104	36.5
	R	384	370	135	55.2	428	386	149	62.9	146	369	120	45.0	195	382	128	47.5
	S	376	371	129	53.0	436	385	145	58.4	136	370	110	39.3	175	387	116	42.0
	TR	365	369	134	56.3	429	390	151	62.1	205	370	120	49.2	255	380	134	52.5
WB	D7	1020	369	144	78.7	1059	410	169	88.1	1269	369	117	57.8	1327	405	128	59.6
	IR	809	368	137	73.4	848	402	160	82.4	652	369	121	60.0	738	380	131	61.0
	IR25	821	368	146	74.0	889	400	169	84.9	656	368	122	59.2	1203	381	143	69.8
	MAD	1053	370	167	93.6	1097	396	189	99.4	1095	370	137	72.4	1051	380	136	63.1
	Gini	1232	369	82.7	32.8	1410	387	91.3	34.5	800	369	68.1	25.6	1059	382	72.3	26.2
	R	1279	370	86.8	34.9	1441	386	94.1	36.8	870	370	74.7	27.7	1082	384	72.1	24.5
	S	1184	369	85.0	34.2	1362	382	93.3	37.2	816	369	67.1	23.5	1229	380	78.6	27.9
	TR	1209	368	83.6	34.2	1433	382	90.9	35.2	699	368	91.9	37.6	850	382	95.7	39.3

Table 3. ARL and SDRL of control charts when diffuse symmetric disturbances are present when m = 30

		n=5								*n*=10							
		ARL				SDRL				ARL				SDRL			
	λ =	0.6	1	1.2	1.4	0.6	1	1.2	1.4	0.6	1	1.2	1.4	0.6	1	1.2	1.4
$\alpha_3 = -1.5$																	
BT	D7	685	381	109	35.8	703	420	131	38.8	646	379	69.0	19.8	668	423	74.5	20.2
	IR	545	377	126	45.6	565	421	156	52.0	257	373	103	31.9	292	404	114	35.3
	IR25	561	386	137	48.8	603	430	170	60.0	264	391	100	32.4	294	405	118	36.1
	MAD	691	377	122	42.3	704	434	140	43.4	411	371	96.5	27.9	482	390	97.9	27.3
	Gini	494	396	96.1	27.0	555	433	112	30.7	147	394	76.0	20.1	178	405	86.8	22.7
	R	510	399	99.2	28.7	564	425	113	32.8	158	375	97.4	27.5	199	399	109	32.3
	S	460	387	95.1	28.0	535	413	112	30.7	143	398	82.7	19.8	173	404	95.5	24.2
	TR	487	404	101	29.4	562	428	116	32.4	197	378	79.4	23.5	235	396	89.4	24.8
$\alpha_3 = -1.0$																	
BT	D7	544	399	97.9	25.2	542	463	122	27.1	381	400	48.2	10.4	389	442	56.9	10.8
	IR	441	408	109	28.7	460	463	152	36.3	141	411	76.3	16.2	150	433	94.8	17.8
	IR25	449	404	124	31.4	457	494	181	42.9	143	390	80.9	17.8	160	432	104	20.4
	MAD	531	408	109	29.2	555	466	136	30.8	213	396	70.3	16.7	231	424	76.0	16.5
	Gini	289	442	91.3	20.0	308	502	124	26.1	54.4	431	70.9	12.1	60.3	442	92.8	14.0
	R	297	452	113	28.0	311	533	164	44.6	58.5	449	153	43.6	67.3	462	231	88.0
	S	272	452	100	21.1	281	484	133	32.0	51.9	439	73.0	10.9	57.1	452	103	13.7
	TR	274	441	100	21.8	299	489	143	31.5	91.4	399	60.5	10.9	102	417	80.3	12.9
$\alpha_3 = 1.0$																	
IG	D7	398	457	171	63.8	414	495	205	80.9	293	469	123	33.2	295	494	149	40.2
	IR	380	423	179	72.7	383	463	218	95.3	117	403	136	44.3	126	416	166	54.7
	IR25	384	415	183	77.8	390	469	234	109	120	399	146	47.2	130	414	175	56.3
	MAD	361	445	202	96.8	380	464	255	127	119	411	163	59.3	127	420	201	75.7
	Gini	120	419	197	73.7	136	416	230	99.9	19.7	323	167	49.8	21.8	360	193	68.4
	R	109	390	216	82.8	127	402	245	109	15.4	240	205	79.7	19.6	287	208	94.1
	S	111	382	199	79.8	128	403	219	106	16.8	268	168	63.3	20.7	307	183	78.2
	TR	108	384	205	81.2	127	391	222	101	57.1	397	126	37.6	63.9	393	147	41.3
WB	D7	491	368	101	40.7	503	405	125	46.7	379	367	69.1	27.8	386	409	87.5	31.8
	IR	424	406	140	54.8	442	435	175	67.2	156	388	110	38.0	169	410	130	45.1
	IR25	449	390	141	58.2	486	444	182	76.2	157	406	118	42.2	171	416	139	49.2
	MAD	460	407	167	82.5	475	450	211	99.5	173	399	151	66.9	189	421	178	76.0
	Gini	198	396	96.1	30.3	226	419	121	39.1	42.2	361	80.2	20.4	49.9	374	94.6	25.2
	R	192	387	105	31.6	224	407	128	39.4	38.8	313	81.7	24.7	50.8	338	93.3	28.9
	S	192	383	93.8	31.1	225	409	112	39.4	38.2	330	72.7	20.3	50.2	368	86.8	25.1
	TR	191	390	97.4	32.2	222	406	116	39.5	85.3	385	78.6	24.1	95.2	396	89.3	25.3
$\alpha_3 = 2.0$																	
IG	D7	559	409	194	100	573	434	223	121	443	400	162	81.0	456	430	192	96.5
	IR	501	376	172	91.4	518	412	204	109	254	361	154	70.6	287	375	165	77.5
	IR25	520	395	188	98.8	561	436	231	122	248	386	161	75.8	285	385	179	85.6
	MAD	538	400	222	127	558	432	247	152	274	370	179	98.3	307	377	200	108
	Gini	296	368	151	66.3	347	384	162	78.6	89.1	330	129	50.2	120	355	139	55.4
	R	286	376	172	76.0	342	393	180	84.5	78.4	301	140	60.8	114	343	146	65.3
	S	288	365	157	69.2	346	366	163	77.1	80.1	310	129	54.7	122	339	136	59.0
	TR	275	370	160	72.0	336	373	171	81.9	166	364	137	55.3	201	374	150	59.9
WB	D7	992	364	148	79.0	1029	415	180	90.2	1254	368	116	57.7	1354	403	125	58.9
	IR	830	367	147	75.6	880	414	174	83.5	602	368	124	60.5	697	380	136	64.2
	IR25	844	371	151	78.9	895	409	176	88.8	658	372	127	60.6	742	383	131	64.6
	MAD	1029	376	167	96.8	1083	408	191	103	1066	364	139	70.1	1202	386	140	69.5
	Gini	1068	374	93.1	38.3	1270	384	102	41.1	604	359	77.6	29.8	853	374	82.8	30.8
	R	1085	380	94.8	38.4	1321	390	103	41.0	625	340	83.3	31.7	1021	366	84.5	31.9
	S	1010	363	95.4	39.4	1249	378	99.7	41.0	557	338	78.1	27.5	838	358	82.2	28.7
	TR	1068	361	90.8	37.3	1336	374	96.6	40.1	651	368	96.5	39.5	806	388	97.0	41.5

Table 4. ARL and SDRL of control charts when diffuse asymmetric variance disturbances are present when m = 30

		n=5								n=10							
		ARL				SDRL				ARL				SDRL			
	λ =	0.6	1	1.2	1.4	0.6	1	1.2	1.4	0.6	1	1.2	1.4	0.6	1	1.2	1.4
$\alpha_3 = -1.5$																	
BT	D7	698	358	102	33.8	712	420	126	36.1	674	367	66.9	19.4	706	396	73.8	19.2
	IR	577	361	116	41.8	575	402	142	49.2	282	364	89.8	30.9	314	403	103	33.9
	IR25	579	373	130	43.9	620	442	173	52.8	292	369	94.3	30.0	327	385	112	33.6
	MAD	703	370	116	40.3	716	412	134	43.4	422	361	90.4	27.3	472	390	97.0	27.3
	Gini	558	376	75.1	24.2	607	406	88.7	27.1	183	351	58.1	16.6	219	369	67.5	18.0
	R	560	358	79.4	24.4	625	402	97.4	27.3	209	361	74.8	21.0	256	377	89.0	23.9
	S	511	357	80.0	23.5	576	395	96.6	26.4	186	360	60.5	15.3	220	382	75.1	16.6
	TR	566	358	84.2	24.1	646	393	100	27.3	224	358	71.3	20.6	251	384	80.8	22.2
$\alpha_3 = -1.0$																	
BT	D7	554	379	85.8	22.6	564	437	109	25.0	422	352	44.5	9.6	422	402	52.7	9.7
	IR	451	359	92.8	24.2	473	445	137	31.4	162	360	64.1	14.2	172	399	80.5	16.1
	IR25	463	379	104	27.9	484	455	162	37.5	161	374	66.5	14.6	179	412	90.7	17.1
	MAD	544	367	96.6	26.9	560	446	116	28.2	242	353	65.8	16.2	266	394	75.3	16.5
	Gini	326	376	62.1	14.4	355	440	91.4	18.8	71.7	371	40.7	8.3	80.0	418	54.8	9.0
	R	341	348	69.4	16.7	363	447	117	27.6	82.1	364	70.4	14.1	90.8	430	128	32.3
	S	331	375	63.5	13.5	349	440	96.5	19.4	74.6	368	38.6	6.9	82.7	393	55.3	7.8
	TR	328	379	66.7	13.8	343	444	103	19.1	104	347	46.0	8.8	117	386	60.6	9.5
$\alpha_3 = 1.0$																	
IG	D7	442	367	118	43.0	440	406	156	51.3	332	368	77.4	23.4	329	407	97.7	27.8
	IR	407	375	138	58.0	431	421	180	71.7	138	359	104	36.0	152	385	126	44.2
	IR25	404	359	150	62.8	424	406	211	85.4	135	368	109	36.9	147	382	141	45.8
	MAD	412	362	154	64.6	415	404	195	89.2	145	369	123	42.3	152	407	152	49.8
	Gini	183	355	99	32.1	195	378	121	36.5	39.4	363	69.5	17.9	44.0	372	81.0	19.5
	R	184	375	115	36.4	206	389	132	42.6	41.2	367	101	28.8	45.7	381	113	30.8
	S	180	371	106	37.0	199	382	124	44.6	40.2	369	82.3	22.0	43.9	377	92.0	23.9
	TR	181	376	113	39.8	194	376	134	45.3	74.4	358	88.8	27.6	79.9	374	106	30.2
WB	D7	490	367	99.6	41.6	503	423	121	48.1	373	358	68.8	27.4	386	411	86.3	30.1
	IR	455	358	121	48.3	482	402	151	60.0	179	360	89.3	33.2	200	387	109	38.9
	IR25	471	360	117	51.2	506	422	159	69.4	167	367	99.1	37.0	185	402	125	43.8
	MAD	488	366	145	70.1	515	412	179	86.9	195	378	124	58.2	213	403	146	66.0
	Gini	263	370	63.5	20.0	270	394	78.0	21.2	63.7	351	43.6	12.6	70.8	373	49.5	13.2
	R	254	375	67.0	21.9	273	396	77.6	22.8	66.4	375	52.1	15.1	76.6	381	58.2	15.9
	S	256	364	63.0	20.4	273	384	73.3	22.0	65.2	369	44.8	11.9	69.9	368	49.1	12.4
	TR	250	362	67.0	21.8	271	381	73.2	23.0	100	372	65.3	20.2	112	380	74.0	21.8
$\alpha_3 = 2.0$																	
IG	D7	576	372	164	87.4	593	406	201	108	513	360	137	68.7	527	399	171	80.2
	IR	548	360	159	81.5	549	392	192	97.8	280	352	135	61.9	304	371	151	71.1
	IR25	545	377	173	88.7	580	413	216	107	278	362	151	67.9	318	376	167	82.5
	MAD	600	369	190	108	608	401	228	125	309	348	158	83.9	341	360	171	94.7
	Gini	372	346	119	48.0	426	359	136	52.7	143	361	97.1	33.1	167	372	110	36.6
	R	368	374	135	55.4	418	394	152	60.9	148	364	123	45.5	198	367	129	46.9
	S	377	348	126	52.0	431	377	136	58.9	130	357	114	41.0	163	364	118	44.5
	TR	373	367	134	56.7	424	372	150	62.7	205	365	124	49.4	241	377	139	54.0
WB	D7	1022	377	143	80.7	1066	426	169	89.2	1279	382	119	56.4	1350	411	130	58.0
	IR	810	356	141	73.3	853	396	165	83.4	612	356	121	60.4	704	383	132	61.6
	IR25	811	370	142	75.6	857	415	172	88.3	662	361	121	58.5	785	367	134	61.3
	MAD	1052	369	169	92.3	1080	404	181	98.9	1056	350	137	69.8	1234	368	145	70.3
	Gini	1208	369	84.4	32.9	1374	375	90.9	35.0	782	362	69.6	24.7	1004	376	71.4	26.0
	R	1195	361	89.8	35.5	1436	381	95.2	37.5	873	373	74.3	27.3	1191	369	78.8	28.4
	S	1208	365	85.2	33.7	1365	378	91.9	35.4	793	365	67.2	24.3	1090	380	72.1	24.8
	TR	1218	347	85.1	33.7	1400	369	92.3	35.2	699	351	90.0	38.2	866	360	95.7	38.7

Table 5. ARL and SDRL of control charts when localized variance disturbances are present for m = 30.

		n=5								n=10							
		ARL				SDRL				ARL				SDRL			
	λ =	0.6	1	1.2	1.4	0.6	1	1.2	1.4	0.6	1	1.2	1.4	0.6	1	1.2	1.4
								$\alpha_3 = -1.5$									
BT	D7	696	346	100	34.2	724	404	123	36.1	699	349	63.9	18.7	735	392	72.6	18.8
	IR	564	374	131	45.2	578	426	161	53.4	265	371	101	33.2	297	392	114	36.0
	IR25	572	375	132	46.8	622	433	168	55.2	276	375	97.9	30.5	317	391	114	34.6
	MAD	697	359	113	40.6	734	405	135	42.9	465	350	84.7	26.3	527	371	92.9	26.0
	Gini	485	396	92.3	28.0	534	428	114	31.6	149	390	77.8	19.8	176	405	89.3	22.3
	R	443	388	93.5	28.3	585	435	113	33.2	159	388	100	27.3	205	379	113	32.2
	S	504	391	94.3	29.7	523	412	109	32.7	141	388	84.8	20.4	181	397	100	23.3
	TR	493	405	97.1	30.3	577	428	115	34.3	174	379	90.8	24.3	209	387	100	26.6
								$\alpha_3 = -1.0$									
BT	D7	561	366	89.0	23.2	571	428	115	25.8	422	357	42.5	9.5	435	405	52.2	9.9
	IR	447	395	116	29.5	464	474	174	39.6	141	392	86.1	17.6	166	420	111	21.5
	IR25	462	389	116	28.9	475	477	172	41.7	152	374	72.8	16.1	169	417	101	19.0
	MAD	549	361	94.2	25.5	567	435	121	27.3	249	324	61.1	15.4	287	374	71.7	15.7
	Gini	300	420	92.4	19.7	324	473	125	25.7	56.6	422	66.2	12.3	64.8	431	91.0	14.6
	R	262	437	102	21.2	324	501	161	41.3	63.7	440	141	33.9	71.8	490	209	74.3
	S	295	437	110	25.4	284	492	136	30.1	53.0	445	75.2	10.9	59.0	453	103	14.0
	TR	272	466	102	21.5	293	504	140	31.2	80.1	410	73.3	12.7	91.5	427	95.6	15.3
								$\alpha_3 = 1.0$									
IG	D7	432	356	119	44.9	439	398	150	54.5	334	350	75.2	22.6	340	397	96.8	26.9
	IR	355	437	216	93.8	377	480	274	137	102	407	184	62.7	119	420	227	86.0
	IR25	381	407	176	73.6	396	451	229	102	120	383	132	43.7	135	398	168	56.2
	MAD	394	402	196	95.6	402	453	256	136	121	409	170	61.5	132	405	207	81.5
	Gini	127	410	198	68.6	150	416	236	99.9	19.7	308	174	47.2	23.5	340	196	64.4
	R	121	382	199	77.9	140	401	224	103	19.0	274	187	70.4	23.9	300	200	86.2
	S	118	379	187	80.4	143	387	215	105	17.1	271	169	59.2	21.6	310	184	77.8
	TR	120	374	205	80.8	142	384	227	108	43.4	365	175	59.5	51.0	374	204	82.0
WB	D7	520	340	88.1	37.5	532	391	114	42.6	399	295	58.1	23.0	412	348	74.1	26.0
	IR	432	376	128	55.8	465	435	176	76.9	178	355	98.5	38.1	196	396	129	47.4
	IR25	476	355	122	50.2	493	418	167	64.4	172	370	98.7	34.9	193	403	128	41.4
	MAD	517	333	133	64.7	519	404	182	79.3	222	316	105	50.1	246	367	131	59.6
	Gini	214	391	94.1	30.1	245	420	117	42.0	45.4	352	75.2	20.4	55.6	373	94.1	26.8
	R	204	380	99.7	33.0	244	411	122	41.6	39.5	316	82.3	24.9	53.7	346	92.8	29.1
	S	198	367	97.1	31.5	240	393	122	40.7	37.4	320	75.9	22.0	51.9	360	89.6	26.6
	TR	193	380	97.1	32.7	236	397	119	42.2	86.5	371	84.1	26.5	100	381	102	32.0
								$\alpha_3 = 2.0$									
IG	D7	605	336	147	78.8	653	381	187	95.8	544	311	109	58.0	578	365	137	69.8
	IR	534	362	173	90.7	550	403	205	119	275	355	140	66.6	315	381	162	78.5
	IR25	582	373	170	89.4	606	409	210	112	293	356	146	68.9	327	379	171	79.2
	MAD	594	357	184	106	607	379	217	130	338	318	141	76.9	382	339	166	89.3
	Gini	318	348	148	63.6	388	367	165	74.5	94.1	328	124	48.2	127	360	137	56.8
	R	306	361	159	72.9	391	372	178	83.2	85.2	303	140	60.0	130	333	145	64.3
	S	300	347	150	70.1	375	362	169	78.6	82.3	284	132	57.7	128	332	140	60.5
	TR	282	352	159	72.5	353	365	169	80.8	166	343	142	61.7	211	357	155	69.1
WB	D7	1090	310	125	68.6	1148	383	147	80.0	1491	298	96.3	51.5	1575	370	108	53.7
	IR	825	332	138	70.7	874	381	160	83.5	717	325	111	54.8	825	364	123	59.3
	IR25	887	333	132	70.0	962	389	164	83.2	741	330	112	56.0	859	364	128	61.8
	MAD	1078	331	149	86.6	1149	380	172	94.3	1315	313	123	65.4	1389	341	129	67.9
	Gini	1135	362	91.2	37.0	1375	381	101	40.2	674	343	76.5	27.5	1002	369	82.1	29.3
	R	1122	364	94.6	39.1	1367	376	100	41.1	625	331	80.7	31.3	1015	350	84.5	32.5
	S	1092	362	94.7	37.5	1364	366	101	39.5	603	331	75.9	28.7	1006	354	80.4	29.9
	TR	1085	337	92.5	37.6	1338	360	96.6	40.7	707	343	92.4	38.9	970	359	95.7	40.8

Table 6. ARL and SDRL of control charts when diffuse mean disturbances are present for m = 30.

		n=5								*n*=10							
		ARL				SDRL				ARL				SDRL			
	λ =	0.6	1	1.2	1.4	0.6	1	1.2	1.4	0.6	1	1.2	1.4	0.6	1	1.2	1.4
$\alpha_3 = -1.5$																	
BT	D7	539	504	167	49.7	572	524	181	50.4	490	470	91.2	22.7	527	473	93.8	22.3
	IR	435	480	202	70.0	447	492	224	77.5	156	419	157	47.5	175	416	162	48.5
	IR25	449	467	202	69.5	484	492	240	80.2	165	416	151	49.7	179	424	157	51.4
	MAD	533	476	173	50.8	557	503	180	52.0	269	417	110	29.9	307	431	111	28.6
	Gini	245	433	168	52.2	262	431	166	53.7	46.6	258	143	47.7	48.8	275	140	48.3
	R	215	394	157	53.0	256	418	166	56.7	39.9	208	146	64.8	44.4	223	148	66.7
	S	238	419	167	55.3	232	395	158	55.4	38.2	228	152	55.8	41.7	242	153	57.1
	TR	237	393	169	55.2	247	402	161	55.9	82.3	328	140	40.8	93.3	351	141	41.9
$\alpha_3 = -1.0$																	
BT	D7	442	549	164	35.6	468	554	184	38.3	316	494	74.1	13.8	324	491	81.6	13.7
	IR	374	533	188	46.1	385	561	240	58.9	99.7	454	135	26.6	109	449	149	29.9
	IR25	368	509	189	46.6	400	534	247	60.9	100	435	133	27.2	111	444	156	31.6
	MAD	446	502	159	35.1	448	531	180	36.3	153	438	96.1	19.0	167	447	99.7	18.8
	Gini	155	449	213	59.3	162	466	209	62.2	20.3	277	197	49.8	20.8	287	201	56.6
	R	142	417	214	67.3	159	448	232	91.3	17.3	199	191	133	18.2	215	196	133
	S	146	444	233	88.3	144	432	216	70.5	17.1	225	216	64.3	17.7	242	213	72.5
	TR	139	419	222	73.1	145	432	224	74.8	50.8	390	140	24.4	54.2	400	155	29.1
$\alpha_3 = 1.0$																	
IG	D7	357	504	237	93.2	380	524	288	119	281	469	140	40.0	278	465	140	39.5
	IR	315	487	259	111	327	503	302	147	89.9	434	217	73.5	95.2	430	248	91.9
	IR25	336	489	244	104	351	513	298	142	88.8	422	212	73	96.3	432	247	90.9
	MAD	322	472	273	126	330	494	330	168	97.1	430	224	83.2	105	436	261	106
	Gini	83.3	399	288	112	85.9	410	297	121	10.9	200	257	82.5	10.7	217	258	87.8
	R	73.9	345	294	126	79.7	354	298	132	8.9	138	228	120	8.6	147	232	123
	S	75.3	361	287	121	76.0	381	281	131	8.8	151	232	97	8.7	150	231	95.1
	TR	73.1	341	296	123	74.6	353	287	130	33.8	331	230	73.7	32.8	332	226	71.4
WB	D7	416	514	194	77.1	421	532	237	97.6	330	455	120	42.5	343	480	154	49.9
	IR	357	491	205	87.9	367	519	243	104	112	432	173	62.4	124	440	198	70.3
	IR25	369	477	205	84.2	389	511	258	112	108	437	175	65.0	120	427	198	74.3
	MAD	383	493	253	119	396	511	284	140	130	452	206	95.0	140	440	220	106
	Gini	111	383	180	55.6	118	396	190	57.1	16.0	208	145	39.8	15.9	227	148	41.4
	R	100	372	188	56.6	106	383	181	59.8	11.6	133	120	45.8	11.5	141	120	44.6
	S	101	368	177	51.9	104	374	180	53.8	12.8	160	126	38.4	13.2	171	125	38.6
	TR	100	352	181	55.0	103	354	175	56.5	44.1	348	147	43.6	48.3	368	153	47.3
$\alpha_3 = 2.0$																	
IG	D7	496	424	233	132	527	463	283	165	450	410	168	85.5	453	398	170	85.3
	IR	382	462	275	152	399	468	305	163	142	381	232	113	155	386	235	125
	IR25	424	462	260	146	449	469	287	174	155	388	234	115	179	394	237	126
	MAD	442	464	286	180	468	473	314	196	193	390	241	139	212	390	245	146
	Gini	150	328	218	110	155	334	221	109	30.4	183	170	82.5	33.1	200	167	81.2
	R	139	326	228	118	150	343	226	122	23.9	140	143	88.9	25.5	157	141	86.1
	S	136	323	214	107	148	326	214	107	23.5	145	142	81.6	23.3	143	140	82.2
	TR	129	297	219	111	142	312	219	110	65.5	264	193	96.0	65.2	261	192	95.9
WB	D7	886	416	188	95.5	938	450	215	106	1176	402	129	60.4	1280	436	141	60.0
	IR	565	470	222	115	615	486	237	124	325	376	162	80.5	377	375	160	81.0
	IR25	629	462	215	112	684	467	226	120	373	390	163	78.3	438	390	164	79.4
	MAD	786	472	225	125	805	492	233	124	674	404	158	71.7	761	415	152	70.0
	Gini	444	369	140	56.7	482	376	139	54.8	130	220	106	45.0	144	232	105	44.7
	R	453	339	134	56.7	479	344	133	56.4	104	172	91.2	42.3	115	187	93.5	42.4
	S	418	336	134	55.3	448	350	135	54.3	101	196	97.8	41.1	116	207	94.0	40.1
	TR	421	337	126	56.2	449	329	125	55.1	210	284	116	53.9	262	303	116	53.5

5. CONCLUSION

Eight estimators of the standard deviation and three skewed distributions were considered in Phase I of the control charting process. We compared estimators, bearing in mind that a good estimator should be efficient against contaminations in the four types of disturbances. The results of the RE scenario simulations showed that Gini performs best, and IR25 is the worst in the first three kinds of small disturbances for the negatively skewed distribution.; D7 has the best efficiencies against the four disturbances in the lowly-skewed distributions, but the performance of TR is the worst compared to the other estimators. The study also discussed the FBP of the eight estimators for various sample sizes. The FBP of D7 and MAD is best, followed by IR25. $\overline{\mathrm{G}}$, $\overline{\mathrm{R}}$, and $\overline{\mathrm{S}}$ have the worst FBP. The Phase II simulations pointed out that, in general, the control charts based on D7, and MAD are most robust, but the performances of those based on Gini, R, or S are the worst in the four disturbances.

Although the study showed that the efficiency and robustness of control charts based on these estimators had been investigated, limitations should be considered in this study. The study only discussed the performance of the eight estimators, which are often used in practice, but some existing estimators of process standard deviation are not considered here, such as M-estimate estimators. Meanwhile, the study does not involve the utilization or comparison of other control charts, such as EWMA or CUSUM. Barring three distributions, BT, IG, and WB, other non-normal distributions such as inverse gamma or Burr still need to be discussed. The limitations should be investigated and eliminated for future research.

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