

A PRACTICAL CONSIDERATION OF THE LEAD TIME DEMAND

Changkyu Park

Department of Management
University of Ulsan
San 29 Mugeo-Dong
Nam-Gu, Ulsan 680-749
Republic of Korea
Email: ckparkuou@ulsan.ac.kr

Due to the importance of lead time demand in the design of inventory management systems, researchers and practitioners have paid continuous attention to it and a few analytic models using the compound distribution approach have been reported. However, since the nature of compound distributions arises a heavy analytic burden, the analytic models have been developed by non-recognition of the compound nature of some components to reduce the analytic task. Through the theoretic examination of the analytic model approach, this paper clarifies the assumptions implicitly made by the analytic models and provides some precautions in using the analytic models. An illustrative example is also presented.

Significance: To researchers and practitioners who attempt to use the analytic models in the design of inventory management systems, this paper provides some precautions through the theoretical investigation of the analytic model approach.

Keywords: Inventory, Lead time demand, Compound stochastic process, Renewal process.

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1. INTRODUCTION

Lead time demand (LTD) has received continuous attention from researchers and practitioners, because the knowledge of the distribution of LTD is essential in the design of inventory management systems. Modeling the distribution of LTD involves taking into account the distributions of both demand and lead time, which is very difficult to obtain. Hence the research on inventory problems is often made under the assumption that LTD follows certain convenient distributions such as the Poisson, Normal, Gamma, Weibull, and so forth (Kumaran and Achary, 1996).

However, some researchers such as Bagchi et al. (1984), Carlson (1982), McFadden (1972), and others explicitly treated the distribution of LTD as a compound distribution. As Bagchi et al. (1984) mentioned, the compound distribution approach has some advantages: (1) the components of the compound distribution can be modeled individually and their parameters estimated; and (2) this is more constructive than attempts to model the compound distribution directly since a single component typically has a simpler structure, and better use can be made of the data.

After reviewing the analytically most satisfying studies, in which successive realizations of period demand and lead time are assumed to be independently and identically distributed random variables and the distribution of LTD is considered as a compound distribution, Bagchi et al. (1984) formalized the approach of obtaining the distribution of LTD (we call it the OS-OI-LT approach and explain in detail the OS-OI-LT approach in section 2). The OS-OI-LT approach first considers order intensity, order size, and lead time as primary components. Then it coalesces two of the primary components into the intermediate component (e.g., demand per unit time, see Figure 2 in section 2), in order to reduce the analytic task in the characterization of LTD.

It is not surprising that the analytically most satisfying studies follow this approach, because, even with only two stochastic components, the analytic determination of the compound distribution is a challenging analytic task. We are now faced with the dilemma that modeling realistic distributions of LTD often causes severe computational difficulties and, more often than not, is analytically intractable, whereas modeling rather simpler distributions of LTD may be inadequate as representations of reality.

Usually we accept, frequently without argument, the derivation of rather simpler distributions as an approximation of reality. However, we need to investigate under what situations the derivation of rather simpler distributions is appropriately performed. The aim of this paper is not to criticize the OS-OI-LT approach, but to assist researchers and practitioners using the OS-OI-LT approach in practice by providing some precautions. First, we define the problem situation considered by this paper and consider possible two approaches of deriving the distribution of LTD as a compound distribution: the OS-OI-LT approach and the OS-IT-LT approach. Then we theoretically examine and compare the two approaches. Based on the

examination and comparison, we scrutinize the characteristics of the OS-OI-LT approach and figure out under what situations the OS-OI-LT approach is appropriately performed. Lastly, we show shows an illustrative example.

2. THE COMPOUND DISTRIBUTION APPROACH TO LEAD TIME DEMAND

The situation we consider can be depicted as Figure 1. Customer orders randomly arrive and we assume that the inter-arrival times between customer orders are independent each other. The size of a customer order measured by the number of units is also randomly varied and we assume that the customer order sizes are independent each other. Then LTD is the total number of units demanded by customers during a lead time (LT). For instance, Figure 1 shows that LTD is 19.

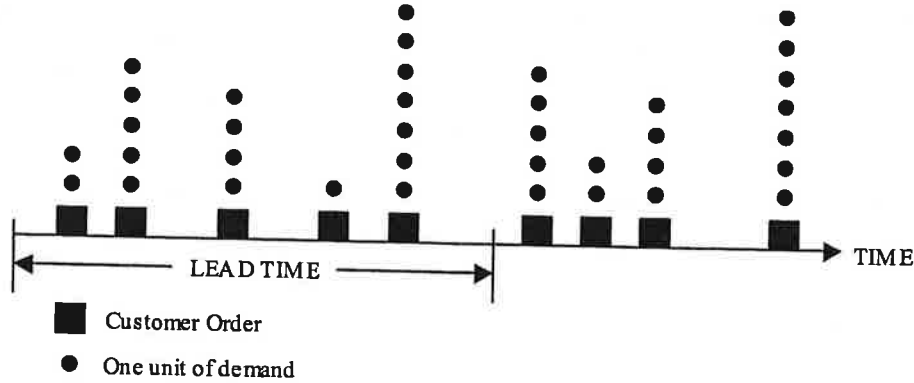


Figure 1. Realization of lead time demand.

If we define $D(t)$ as the cumulative demand during the time interval $(0, t)$, $0 \leq t$, then LTD can be described as $D(LT)$, where LT or replenishment time, as related to inventory management systems, is the time interval between placing a replenishment order and receiving it from the supplier and $LT \geq 0$.

Approaches to generating LTD can be depicted as Figure 2. It shows two approaches to deriving LTD. The one approach on the left hand side is the OS-OI-LT approach formalized by Bagchi et al. (1984). It first derives demand per unit time (DPUT), which is the total demand in a period, by combining order intensity (OI), the number of orders per period, and order size (OS), the size of each order. Then LTD is generated by joining DPUT and LT . (Bagchi et al. (1984) formalized another approach of using the concept of lead time order intensity (LTOI), the total number of orders during a lead time, which is derived by combining OI and LT . However, we omit this approach in this paper because this approach is basically same as the OS-OI-LT approach and is not much known in the literature.)

The other approach on the right hand side is the OS-IT-LT approach. The OS-IT-LT approach first derives $D(t)$ using information of OS and inter-arrival time (IT), the time duration between consecutive customer orders. Then LTD is generated by joining $D(t)$ and LT . The OS-IT-LT approach is more appropriate to the situation we consider than the OS-OI-LT approach. However, the OS-IT-LT approach requires some computational efforts in deriving LTD. We utilize the OS-IT-LT approach to investigate the characteristics of the OS-OI-LT approach.

2.1. The OS-IT-LT Approach

This approach derives the total demand during a time interval $(0, t)$, $D(t)$, using information of OS and IT. Suppose a sequence of order size, $\{OS_i, i = 0, 1, 2, \dots\}$, in which each OS_i is independent and follows an identical probability mass function of $h(\cdot)$ and cumulative distribution function of $H(\cdot)$ having mean, μ_{OS} and variance, σ^2_{OS} . Then the total demand during a time interval $(0, t)$ can be defined as follows. For $t \geq 0$,

$$D(t) = OS_0 + OS_1 + \dots + OS_{N(t)} = \sum_{i=0}^{N(t)} OS_i \quad \dots \quad (1)$$

where $N(t)$ is the number of customer orders during a time interval $(0, t)$, $N(0) = 0$, and $D(0) = 0$.

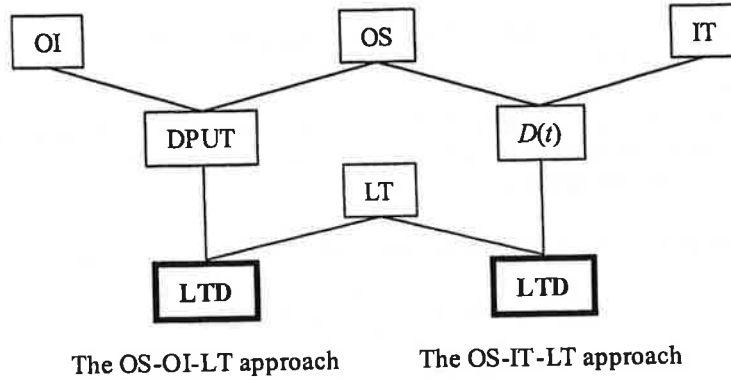


Figure 2. Approaches to lead time demand.

Let $F(d; t)$ be a cumulative distribution function of $D(t)$, given the time t . Then for $d, t \geq 0$,

$$\begin{aligned}
 F(d; t) &= \Pr\{D(t) \leq d\} = \Pr\left\{\sum_{i=0}^{N(t)} OS_i \leq d\right\} \\
 &= \sum_{n=0}^{\infty} \Pr\left\{\sum_{i=0}^n OS_i \leq d \mid N(t) = n\right\} \Pr\{N(t) = n\} \\
 &= \sum_{n=0}^{\infty} \Pr\left\{\sum_{i=0}^n OS_i \leq d\right\} \Pr\{N(t) = n\} \quad \dots \quad (2)
 \end{aligned}$$

Here, suppose the successive occurrence times between customer orders, $\{IT_i, i = 0, 1, 2, \dots\}$, in which each IT_i is independent and follows an identical probability density function of $g(\cdot)$ and cumulative distribution function of $G(\cdot)$ having mean, μ_{IT} and variance, σ_{IT}^2 . Then a nonnegative integer-valued stochastic process $\{N(t), t \geq 0\}$ is a renewal process that registers the successive occurrences of a customer order during the time interval $(0, t)$.

Let W_n be a waiting time until the occurrence of the n th customer order. Then for $n \geq 0$,

$$W_n = \sum_{i=0}^n IT_i \quad \dots \quad (3)$$

where $W_0 = 0$ and $IT_0 = 0$. The fundamental connecting link between the waiting time process $\{W_n, n = 0, 1, 2, \dots\}$ and the renewal process $\{N(t), t \geq 0\}$ is the observation that $N(t) \geq n$ if and only if $W_n \leq t$ (Taylor and Karlin, 1984). It follows that for $n, t \geq 0$,

$$\Pr\{N(t) \geq n\} = \Pr\{W_n \leq t\} = \Pr\left\{\sum_{i=0}^n IT_i \leq t\right\} = G_n(t) \quad \dots \quad (4)$$

where $G_n(t)$ is the n -fold convolution of $G(t)$, such that for $t \geq 0$,

$$G_n(t) = \int_0^t G_{n-1}(t-y)dG(y) = \int_0^t G_{n-1}(t-y)dG(y) \quad n \geq 2 \quad \dots \quad (5)$$

Here, $G_0(t) = 1$ and $G_1(t) = G(t)$.

Consequently, from the equation (4), for $n, t \geq 0$,

$$z(n; t) = \Pr\{N(t) = n\} = \Pr\{N(t) \geq n\} - \Pr\{N(t) \geq n+1\} = G_n(t) - G_{n+1}(t) \quad \dots \quad (6)$$

By substituting the equation (6) into the equation (2), for $d, t \geq 0$,

$$F(d; t) = \sum_{n=0}^{\infty} H_n(d)z(n; t) \quad \dots \quad (7)$$

where $H_n(d)$ is the n -fold convolution of $H(d)$, such that for $d \geq 0$,

$$H_n(d) = \sum_{y=0}^{\infty} H_{n-1}(d-y)h(y) = \sum_{y=0}^d H_{n-1}(d-y)h(y) \quad n \geq 2 \quad \dots \quad (8)$$

Here, $H_0(d) = 1$ and $H_1(d) = H(d)$.

The cumulative distribution function of $D(t)$ described as equation (7) is determined, in principle, by equations (5) and (8). A more detailed characterization of $D(t)$ can be obtained by transform methods. We present Lemmas 1 and 2 for general discrete and continuous distributions, respectively.

Lemma 1. If $h(x)$ is a (discrete) probability mass function for $x = 0, 1, 2, \dots$, then $H_n(x)$ is defined as, for $n \geq 1$,

$$H_n(x) = \sum_{k=0}^x \tilde{h}_n(k) \quad \dots \quad (9)$$

where

$$\tilde{h}_n(k) = \sum_{\substack{j_0+\dots+j_k=n \\ j_1+2j_2+\dots+kj_k=k}} \frac{n!}{\prod_{l=0}^k j_l!} \prod_{l=0}^k h(l)^{j_l} \quad \dots \quad (10)$$

and j_l is a nonnegative integer.

Proof. Using the discrete Laplace operator $L[u] = \sum_{k=0}^{\infty} u(k)s^k$, we obtain

$$\begin{aligned} L[H_n] &= L[H_{n-1} * h] = L[H_{n-1}]L[h] = L[1]L[h]^n \\ &= \frac{1}{1-s} \left(\sum_{k=0}^{\infty} h(k)s^k \right)^n = \frac{1}{1-s} \sum_{k=0}^{\infty} \tilde{h}_n(k)s^k = \sum_{k=0}^{\infty} \tilde{h}_n(k) \frac{1}{1-s} s^k \quad \dots \quad (11) \end{aligned}$$

where $*$ is the discrete finite convolution and $\tilde{h}_n(k)$ defined as equation (10) is the coefficient of the term s^k in $\left(\sum_{j=0}^k h(j)s^j \right)^n$.

Defining $\delta(x=0)$ as an indicator function, that is, 1 if $x = 0$, and 0, otherwise, and applying the inverse Laplace operator L^{-1} to the equation (11), we obtain

$$\begin{aligned} H_n(x) &= \sum_{k=0}^{\infty} \tilde{h}_n(k)L^{-1}\{L[1]L[\delta(x-k=0)]\} = \sum_{k=0}^{\infty} \tilde{h}_n(k)L^{-1}\{1 * \delta(x-k=0)\}(x) \\ &= \sum_{k=0}^{\infty} \tilde{h}_n(k) \sum_{l=0}^x \delta(l-k=0) = \sum_{k=0}^x \tilde{h}_n(k) \quad \dots \quad (12) \end{aligned}$$

Lemma 2. If $g(t)$ is a (continuous) probability density function for $t \geq 0$ and $\int_0^a g(t)dt \cong 1.0$ then $G_n(t)$ is approximately defined as, for $0 \leq t \leq a$

$$G_n(t) = t^n \sum_{k=0}^{nl} b_n(k) \frac{1}{(k+n)!} t^k \quad \dots \quad (13)$$

where l is the number of equal interval dividing the time period of $0 \leq t \leq a$ and

$$b_n(k) = \sum_{\substack{j_0+\dots+j_k=n \\ j_1+2j_2+\dots+kj_k=k}} \frac{n!}{\prod_{l=0}^k j_l!} (l!a_l)^{j_l} \quad \dots \quad (14)$$

j_l is a nonnegative integer

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_l \end{bmatrix} = \begin{bmatrix} 1 & t_0 & t_0^2 & \dots & t_0^l \\ 1 & t_1 & t_1^2 & \dots & t_1^l \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_l & t_l^2 & \dots & t_l^l \end{bmatrix}^{-1} \begin{bmatrix} g(t_0) \\ g(t_1) \\ \vdots \\ g(t_l) \end{bmatrix}$$

$$t_i = i \frac{a}{l} \quad (0 \leq i \leq l)$$

Proof. We approximate $g(t)$ by a Lagrange interpolating polynomial $p_l(t)$ at points $(t_i, g(t_i))$ with $t_i = i \frac{a}{l}$ ($0 \leq i \leq l$). Then $G_n(t)$ in the equation (5) is redefined as

$$G_n(t) = \int_0^t G_{n-1}(t-y)p_l(y)dy \quad \dots \quad (15)$$

Applying the continuous Laplace operator $L[u] = \int_0^\infty u(t)e^{-st} dt$ to the equation (15), we obtain

$$\begin{aligned} L[G_n] &= L[G_{n-1} * p_l] = L[G_{n-1}]L[p_l] = L[1]L[p_l]^n \\ &= \frac{1}{s} \left(\int_0^\infty p_l(t)e^{-st} dt \right)^n = \frac{1}{s} \left(\int_0^\infty \sum_{k=0}^l a_k t^k e^{-st} dt \right)^n = \frac{1}{s} \left(\sum_{k=0}^l a_k \int_0^\infty t^k e^{-st} dt \right)^n \\ &= \frac{1}{s} \left(\sum_{k=0}^l a_k \frac{k!}{s^{k+1}} \right)^n = \frac{1}{s^{1+n}} \left(\sum_{k=0}^l a_k \frac{k!}{s^k} \right)^n \equiv \frac{1}{s^{1+n}} \sum_{k=0}^{nl} b_n(k) \frac{1}{s^k} \\ &= \sum_{k=0}^{nl} b_n(k) \frac{1}{s^{k+n+1}} = \sum_{k=0}^{nl} b_n(k) L[t^{k+n}] \quad \dots \quad (16) \end{aligned}$$

where $b_n(k)$ defined as equation (14) is the coefficient of the term $\frac{1}{s^j}$ in $\left(\sum_{j=0}^k a_j \frac{j!}{s^j} \right)^n$. Finally, applying the inverse Laplace operator L^{-1} to the equation (16), we obtain the equation (13).

Lastly, LTD is generated by joining $D(t)$ and LT. Let $L(d)$ be a cumulative distribution function of LTD and LT follow a probability density function of $k(\cdot)$ having mean, μ_{LT} and variance, σ_{LT}^2 . Then, for $d \geq 0$,

$$L(d) = \sum_{x=0}^d \int_0^\infty \Pr\{D(t) = x\}k(t)dt = \sum_{x=0}^d \int_0^\infty [F(d;t) - F(d-1;t)]k(t)dt \quad \dots \quad (17)$$

2.2. The OS-OI-LT Approach

This approach reduces the burden of analytic task in determining LTD by implicitly dealing with the compound nature of DPUT, resulted from coalescing two components of OS and OI. Suppose that the order intensity, OI, follows a probability mass function of $p(\cdot)$ having mean, μ_{OI} and variance, σ_{OI}^2 . Then the demand per unit time can be defined as

$$DPUT = OS_0 + OS_1 + \dots + OS_{OI} = \sum_{i=0}^{OI} OS_i \quad \dots \quad (18)$$

and an associated probability distribution can be described as follows. For $d \geq 0$,

$$\begin{aligned} \Pr\{DPUT \leq d\} &= \Pr\left\{ \sum_{i=0}^{OI} OS_i \leq d \right\} \\ &= \sum_{n=0}^\infty \Pr\left\{ \sum_{i=0}^{OI} OS_i \leq d \mid OI = n \right\} \Pr\{OI = n\} = \sum_{n=0}^\infty \Pr\left\{ \sum_{i=0}^n OS_i \leq d \right\} p(n) \\ &= \sum_{n=0}^\infty H_n(d)p(n) \quad \dots \quad (19) \end{aligned}$$

In order to derive LTD, the OS-OI-LT approach combines DPUT and LT. However, the equation (19) does not contain the parameter of time. Hence, in the OS-OI-LT approach, the total demand during a time interval $(0, t)$, $D(t)$, is implicitly

derived from DPUT by proportionally including a time duration. In other words, since DPUT is a demand per unit time, the OS-OI-LT approach assumes that $D(t)$ is formulated by involving a time proportionally on DPUT. Thus it follows that, for $d \geq 0$ and $t > 0$,

$$\Pr\{D(t) \leq d\} = \Pr\{DPUT * t \leq d\} = \Pr\{DPUT \leq \frac{d}{t}\} = \sum_{n=0}^{\infty} H_n \left(\frac{d}{t}\right) p(n) \quad \dots \quad (20)$$

Compared to equation (7), the equation (20) shows that the OS-OI-LT approach reduces the burden of analytic task by not recognizing the compound nature of DPUT. Lastly, LTD is similarly derived as equation (17).

3. COMPARISON OF TWO APPROACHES

Bagchi et al. (1984) said that non-recognition of the compound nature of the intermediate component in the OS-OI-LT approach is a price that may have to be paid in order to obtain an analytic solution. If, however, the price is unaffordable, we think, it would be better to consider alternative approaches. By comparing the two approaches presented in section 2, this section examines the characteristics of demand processes and figures out under what situations the OS-OI-LT approach is appropriately performed.

In the OS-IT-LT approach, the total demand during a time interval $(0, t)$, $D(t)$, is a random sum of random variables since OS_i and $N(t)$ in the equation (1) are random variables. Mean and variance of $D(t)$ can be derived as (see Taylor and Karlin (1984))

$$\begin{aligned} E[D(t)] &= \mu_{OS} E[N(t)] \\ Var[D(t)] &= E[N(t)] \sigma_{OS}^2 + \mu_{OS}^2 Var[N(t)] \quad \dots \quad (21) \end{aligned}$$

In the same token, from the equation (18), DPUT is also a random sum of random variables. Mean and variance of DPUT can be derived as

$$\begin{aligned} E[DPUT] &= \mu_{OS} \mu_{OI} \\ Var[DPUT] &= \mu_{OI} \sigma_{OS}^2 + \mu_{OS}^2 \sigma_{OI}^2 \quad \dots \quad (22) \end{aligned}$$

Since the OS-OI-LT approach assumes that $D(t)$ is formulated by involving a time proportionally on DPUT, mean and variance of $D(t)$ for the OS-OI-LT approach can be derived as

$$\begin{aligned} E[D(t)] &= \mu_{OS} \mu_{OI} t \\ Var[D(t)] &= (\mu_{OI} \sigma_{OS}^2 + \mu_{OS}^2 \sigma_{OI}^2) t \quad \dots \quad (23) \end{aligned}$$

We use the variance-to-mean ratio (VMR) to measure demand variability. The VMR (also called the index of dispersion), though less well known than the often-used coefficient of variation, is of value in providing the discrimination for discrete distributions (Ord, 1973). According to equation (23), the OS-OI-LT approach is based on the assumption that the VMR of the cumulative demand is constant. In other words, mean and variance of the cumulative demand increase proportionally to the period of time with the same rate. This assumption can be justified when the renewal process $\{N(t), t \geq 0\}$ in the equation (21) is the Poisson process. In other words, the OS-OI-LT approach is compatible with the OS-IT-LT approach only when IT follows the exponential distribution.

However, according to the asymptotic behaviors of $N(t)$ such that (see Taylor and Karlin (1984))

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{E[N(t)]}{t} &= \frac{1}{\mu_{IT}} \\ \lim_{t \rightarrow \infty} \frac{Var[N(t)]}{t} &= \frac{\sigma_{IT}^2}{\mu_{IT}^3} \quad \dots \quad (24) \end{aligned}$$

the asymptotic mean and variance of $N(t)$ in the equation (21) are approximately t/μ_{IT} and $t\sigma_{IT}^2/\mu_{IT}^3$, respectively. This means that the two approaches generate the similar results for $D(t)$, no matter whatever distribution IT follows, provided the considered period of time is large enough.

Based on the theoretical examination described so far, we can see that the OS-OI-LT approach is mostly suitable for the situation that the demand process is a compound Poisson process. We can also speculate that, even for the other situations,

the OS-OI-LT approach is applicable when mean of LT, μ_{LT} is sufficiently large to permit use of the asymptote. We demonstrate this speculation with an illustrative example in the next section.

4. AN ILLUSTRATIVE EXAMPLE

In order to avoid the impact of special distributions in an illustrative example, we consider the situation that customer orders are randomly arrived between 1.0 and 9.0 (that is, $\mu_{IT} = 5.0$) and the customer order sizes are also randomly varied between 15 and 25. In order to see how the OS-OI-LT approach is affected by the time period of LT, we consider exponential distributions with $\mu_{LT} = 3.0 (\leq \mu_{IT})$ and $\mu_{LT} = 10.0 (\geq \mu_{IT})$ as distributions of LT. Using equation (21), mean, variance, and VMR of $D(t)$ are calculated as shown in Figure 3. We can see that the VMR of $D(t)$ approaches some constant value as the period of time increases.

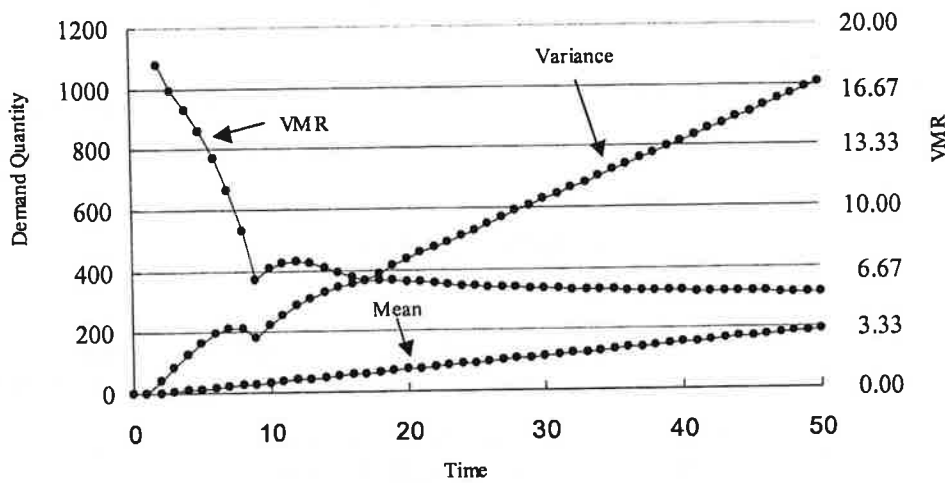


Figure 3. Mean, variance, and VMR of $D(t)$.

The cumulative distribution function of LTD, $L(d)$, which we concern, can be obtained from the equation (17) for the OS-IT-LT approach. However, in order to obtain $L(d)$ (resulted from $D(t)$ in equation (20)) for the OS-OI-LT approach, we need information about OI, $p(n)$ which is hardly determined analytically. Thus, we used Gamma distribution as an asymptotic approximation for $D(t)$ because our separated experiment results showed that given the time t , $D(t)$ was skewed to the right and that data plotting visually confirmed it. Here Gamma distribution has the parameter α depending on the time t since the OS-OI-LT approach is based on the assumption that the VMR of the cumulative demand is constant (see equation (23)).

Figures 4 and 5 show comparisons of $L(d)$ from the two approaches for $\mu_{LT} = 3.0$ and $\mu_{LT} = 10.0$, respectively. From these comparisons, we can see that the results around an asymptotic mean of 12 ($= \mu_{OS} * \mu_{LT} / \mu_{IT}$) generated by the OS-OI-LT approach when $\mu_{LT} < \mu_{IT}$ are worse than the results around an asymptotic mean of 40 when $\mu_{LT} > \mu_{IT}$. We can expect that the OS-OI-LT approach will show better result as $\mu_{LT} \gg \mu_{IT}$.

5. CONCLUSION

The derivation of lead time demand distribution involves the consideration of both demand and lead time, resulted in the compound distribution, which is a challenging analytic task. In order to reduce the analytic task, the analytically most successful studies were done by non-recognition of the compound nature of some components. This paper theoretically investigated the mostly used approach, i.e., the OS-OI-LT approach formalized Bagchi et al. (1984) and clarified the assumption implicitly made by the OS-OI-LT approach.

Comparing the OS-OI-LT approach with the OS-IT-LT approach, we could figure out that (1) the OS-OI-LT approach is mostly suitable for the situation that the demand process is a compound Poisson process; and (2) even for the other

situations, the OS-OI-LT approach is applicable when the mean of lead time is sufficiently large to permit use of the asymptote. However, as shown by an illustrative example, if the mean lead time is not large, the result of the OS-OI-LT approach are truly questionable. Researchers and practitioners who attempt to use the OS-OI-LT approach in practice should be warned on this situation, and the future study is required on this theme.

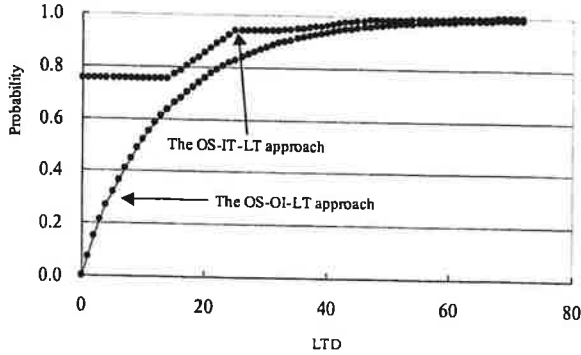


Figure 4. The cumulative distribution of LTD when $\mu_{LT} < \mu_{OT}$.

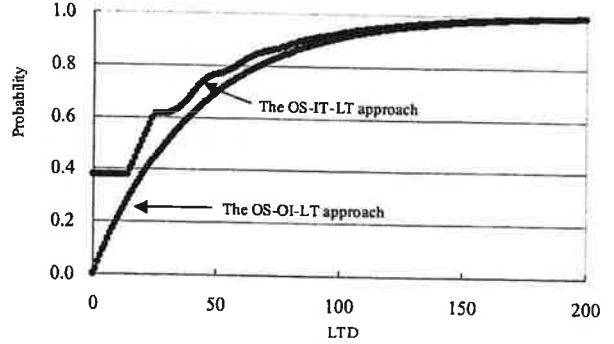


Figure 5. The cumulative distribution of LTD when $\mu_{LT} > \mu_{OT}$.

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APPENDIX

In section 2, we presented Lemmas 1 and 2 for general discrete and continuous distributions, respectively. However, these Lemmas require heavy computational efforts since they are formulated to cover any arbitrary distributions. In order to reduce computational efforts in an illustrative example described in section 4, we used Lemmas A1 and A2 for discrete and continuous uniform distributions, respectively.

Lemma A1. If $h(x)$ follows a discrete uniform distribution with $(a+1 \leq x \leq b)$, then $H_n(x)$ is defined as, for $n \geq 1$,

$$H_n(x) = \frac{1}{(b-a)^n n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \sum_{l=0}^x \left(\prod_{i=0}^{n-1} (x-l-i) \right) \delta(l - an - (b-a)j = 0) \quad \dots \quad (A1)$$

where $\delta(x=0)$ is an indicator function, that is, 1 if $x=0$, and 0, otherwise.

Proof. Note that

$$\begin{aligned} L[H_1] &= \left(\frac{1}{b-a} s^{a+1} + \frac{2}{b-a} s^{a+2} + \dots + \frac{b-a}{b-a} s^b \right) + s^{b+1} + \dots \\ &= (1 + s + s^2 + s^3 + \dots) \frac{1}{b-a} (s^{a+1} + s^{a+2} + \dots + s^b) \\ &= \frac{1}{1-s} L[h] = L[1]L[h] \quad \dots \quad (A2) \end{aligned}$$

Applying the discrete Laplace operator L and using equation (A2), we obtain

$$\begin{aligned}
 L[H_n] &= L[H_{n-1} * h] = L[H_{n-1}]L[h] = L[1]L[h]^n \\
 &= \frac{1}{1-s} \left(\sum_{k=0}^{\infty} h(k)s^k \right)^n = \frac{1}{1-s} \left(\sum_{k=a+1}^b \frac{1}{b-a} s^k \right)^n \\
 &= \frac{1}{(b-a)^n} \frac{s^{(a+1)n} (1-s)^{b-a} n}{(1-s)^{n+1}} = \frac{1}{(b-a)^n} \frac{s^n}{(1-s)^{n+1}} s^{an} (1-s)^{b-a} n \\
 &= \frac{1}{(b-a)^n} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{s^n}{(1-s)^{n+1}} s^{an+(b-a)j} \\
 &= \frac{1}{(b-a)^n} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{n!} L[k \mathbf{P}_n] L[\delta(k - an - (b-a)j = 0)] \quad \dots \quad (A3)
 \end{aligned}$$

where $k \mathbf{P}_n = \prod_{j=0}^{n-1} (k - j)$. Finally, applying the inverse Laplace operator L^{-1} to the equation (A3), we obtain the equation (A1).

Lemma A2. If $g(t)$ follows a continuous uniform distribution with $(0 \leq a \leq t \leq b)$, then $G_n(t)$ is defined as, for $n \geq 1$,

$$G_n(t) = \frac{1}{(a-b)^n} \sum_{l=0}^n \frac{(-1)^{n-l} \binom{n}{l}}{n!} \tilde{G}_l(t) \quad \dots \quad (A4)$$

where

$$\tilde{G}_l(t) = \begin{cases} 0 & t \leq (b-a)l + an \\ (t - (b-a)l - an)^n & (b-a)l + an \leq t \end{cases}$$

Proof. Using the continuous Laplace operator L , we obtain

$$\begin{aligned}
 L[G_n] &= L[G_{n-1} * g] = L[G_{n-1}]L[g] = L[1]L[g]^n \\
 &= \frac{1}{s} \left(\int_0^{\infty} g(t)e^{-st} dt \right)^n = \frac{1}{s} \left(\int_a^b \frac{1}{b-a} e^{-st} dt \right)^n \\
 &= \frac{1}{s} \frac{1}{(b-a)^n} \left(-\frac{1}{s} (e^{-sb} - e^{-sa}) \right)^n \\
 &= \frac{1}{(a-b)^n} \frac{1}{s^{n+1}} \sum_{l=0}^n \binom{n}{l} (e^{-sb})^l (e^{-sa})^{n-l} \\
 &= \frac{1}{(a-b)^n} \sum_{l=0}^n \frac{(-1)^{n-l} \binom{n}{l}}{s^{n+1}} e^{-s\{(b-a)l+an\}} \quad \dots \quad (A5)
 \end{aligned}$$

Finally, applying the inverse Laplace operator L^{-1} to the equation (A5), we obtain the equation (A4).