International Journal of Industrial Engineering, 30(6), 1525-1539, 2023

BAYESIAN INFERENCE TO ESTIMATE RANDOM FAILURE PROBABILITY

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This study introduces a process based on Bayesian inference, enhancing the accuracy of random failure probability estimation, outlined in a detailed six-step procedure. This method focuses on comprehensive data analysis and precise probability estimations, proving particularly beneficial for limited datasets. Applied to brake disc random failure probability assessment, our approach's results were compared with those obtained through Maximum Likelihood Estimation (MLE) across various specimen sizes. This comparative analysis included both graphical and statistical evaluations. The experimental findings demonstrate that our Bayesian inference-based process effectively addresses the challenges posed by small datasets, significantly enhancing estimation accuracy. This methodology is especially advantageous in scenarios where data collection is difficult, providing reliability engineers with an essential framework for leveraging prior information to improve risk management in diverse industrial applications.

Keywords: Reliability Engineering; Reliability Test; Random Failure; Statistical Analysis; Bayesian Inference

(Received on August 15, 2023; Accepted on December 14, 2023)

1. INTRODUCTION

In light of the Fourth Industrial Revolution's progression, intensified competition among automotive and semiconductor manufacturers accentuates the imperative of augmenting component reliability. The mandate for reliability remains paramount, as an unforeseen failure might culminate in the loss of human lives (Chambel *et al*., 2023). The elucidation of failure mechanisms becomes indispensable, given the intrinsic association between product reliability, safety, and adherence to national standards.

Historically, product lifetime tests often applied stresses and used the resulting data to estimate the shape parameter of the Weibull distribution that is commonly employed to assess the rates of various failures (Roesch, 2012; Sim *et al*., 2022). While tests focusing on failure modes, mechanisms, and factors impacting reliability are available, along with general accelerated test methods (Davila-Frias *et al*., 2020), traditional reliability test theorists commonly assume that statistical inference effectively estimates the probability of failure over time. Thus, the main emphasis is on wear-out failures.

Existing studies mainly addressed reliability analysis of systems related to random failures in the reliability engineering field. For instance, A method was studied that short of destruction was detected in the large power system (Ryu and Chang, 2005). The random failures have aroused from independent among components to estimate a phased mission system (Huang *et al*., 2019). Xing *et al*. (2019) studied a combinatorial reliability model for correlated systems, probabilistic competitions and random failure propagation time for dependent components. When Reed *et al*. (2019) cited Coolen and Coolen-Maturi, they said that the survival signature is similar to the system signature and fulfills the role of a quantitative model of the system reliability structure, which is entirely separated from the random failure times of the components. Recently, random failure thresholds have been studied. Xia *et al*. (2022) addressed random failure thresholds and competing failure model in micro-electro-mechanical system. Wang *et al*. (2023) addressed an analytical expression of the remaining useful life (RUL) distribution of aero-engines based on the random-coefficient regression model considering the random failure threshold.

In contrast, we have proposed a novel test and statistical methodology to investigate random failures, as delineated by Sim *et al*. (2022). Our approach employs a plot where the horizontal axis represents stress levels at which random failures occur, and the vertical axis depicts failure characteristics. This method deviates from traditional approaches that estimate failure probability based on time or cycles. Instead, it utilizes stress levels for estimation, thereby addressing the dynamic nature of environmental stresses. We applied statistical analysis to experimental data for estimating failure probabilities under varying environmental conditions. These tests, alongside the new statistical methods using Maximum Likelihood Estimation (MLE), identified hot judder as the predominant cause of random brake disc failures. Our statistical method enables the

DOI: 10.23055/ijietap.2023.30.6.9595 ISSN 1943-670X

estimation of random failure probabilities across a continuous range of stress levels. This is in contrast to conventional methods employing an exponential distribution, which are limited by assuming a constant failure rate. Therefore, our method, leveraging stress levels for estimations, reflects the probabilities of random failures more accurately in fluctuating environmental stress conditions.

Sim *et al*. (2022) assumed sufficient test data was available to estimate the probabilities of random failure. In this reason, the probabilities of random failure were estimated by MLE. A limitation of frequency statistics, specifically when utilizing MLE, is the reduced accuracy of estimation due to the requirement for a larger data set (Wang *et al*., 2009). Therefore, practical issues such as insufficient testing equipment and the number of specimens need to be overcome for this study.

To address these issues, we selected Bayesian inference to estimate probabilities of insufficient data. It is four main groups to apply Bayesian inference in the reliability engineering field. There are merging field and laboratory data to enhance the precision of model parameters, deriving reliability measurements, estimating the RUL, and encompass miscellaneous studies respectively. Detailed information is described in Section 2.

The aim of this study is to develop a methodology for estimating the probability of random failure using Bayesian inference, especially in cases where the dataset is constrained due to limited test equipment or specimen count. Accordingly, the statistical analysis of random failure encompasses a six-step process designed to refine the accuracy of failure probability estimations. Posterior distributions are estimated using field data or prior information, thereby enabling precise parameter estimations for probability distributions. Trajectory models are formulated by generating random numbers based on these estimated probability distributions across various stress levels.

Section 2 reviews related works. Section 3 introduces the Bayesian inference theory and an earlier estimation of random failure probabilities (Sim *et al*., 2022) and shows how to estimate these probabilities using Bayesian inference. Section 4 describes the numerical experiments that support the proposed process. Section 5 contains the conclusions and plans for further work.

2. LITERATURE REVIEW

Research in reliability engineering utilizing Bayesian inference can be categorized into four main groups. The first group encompasses studies merging field and laboratory data, aiming to enhance the precision of model parameters. For instance, Wang *et al*. (2017) utilized real-world wind turbine gearbox data to accurately estimate reliability characteristics. Similarly, Kim *et al*. (2021) adopted an adaptive approach for pipeline corrosion assessment, leveraging both field and laboratory data to refine defect predictions.

The second category is centered on deriving reliability measurements. Common cause failure rates were estimated by Nguyen and Gouno (2020) through causal inference in scenarios with missing data, linking the failure rates of system components to occurrences of both component and system failures. Cheng and Lu (2021) employed an adaptive Bayesian support vector regression model for structural reliability exploration. They combined regression with Monte Carlo simulation to assess the reliability of intricate structures with limited data, adaptively modifying the kernel parameter to counteract overfitting, thereby enhancing the reliability analysis's accuracy and efficiency. Mun *et al*. (2019) employed accelerated life tests (ALTs) to assess the reliability of one-shot devices, addressing the challenges posed by incomplete lifetime data. Using a Bayesian framework, the study introduced three distinct priors for the parameters of the Weibull distribution. Simulations were conducted using Gibbs sampling, and it was demonstrated that the convergence of Bayesian estimates improved as the specimen size increased.

The focus of the third group lies in estimating the RUL. Di and Shaoping (2018) applied a Bayesian model averaging (BMA) method for monotonic degradation data reliability analysis using inverse Gaussian and Gamma processes. Their approach delineated the product degradation path and RUL based on a degradation dataset and was benchmarked against other techniques using both simulated and real mechanical bearing test data. Pang *et al*. (2021) estimated equipment RULs by integrating data from accelerated degradation tests and condition monitoring, presenting a model that considers both degradation trajectories and current equipment states. Their case study on a gearbox estimated its RUL by amalgamating data from accelerated tests and vibration signals from a condition monitoring system. In similar study, Davoudpour (2019) introduced a Bayesian network to assess the impact of maintenance strategies on the reliability and costs of wind turbines. The study underscored the significance of enhancing reliability to mitigate Operating and Maintenance (O&M) expenses.

The fourth category includes a diverse range of studies. Leoni *et al*. (2021) investigated the impact of prior choice, a frequent source of uncertainty in Bayesian analyses. Shuto and Amemiya (2022) employed a sequential Bayesian inference method to estimate Weibull distribution parameters, commonly used for modeling failure rates and to assess system reliability. Their approach began with an initial hyperparameter optimization step to enhance estimation accuracy, with a case study centered on a turbine blade's reliability.

Our work aligns with the first category, employing a Bayesian inference-based process for estimating random failure probabilities.

3. USE OF BAYESIAN INFERENCE TO ESTIMATE RANDOM FAILURE PROBABILITY

3.1 Designing Test and MLE Method Used to Estimate Random Failure (Sim *et al*., 2022)

To effectively employ Bayesian inference, test data are essential as this method relies on estimating parameters based on prior information. The subsequent sections detail a test and data analysis conducted using the MLE method. This approach is depicted in Figure 1, which illustrates the process of deriving random failure probabilities.

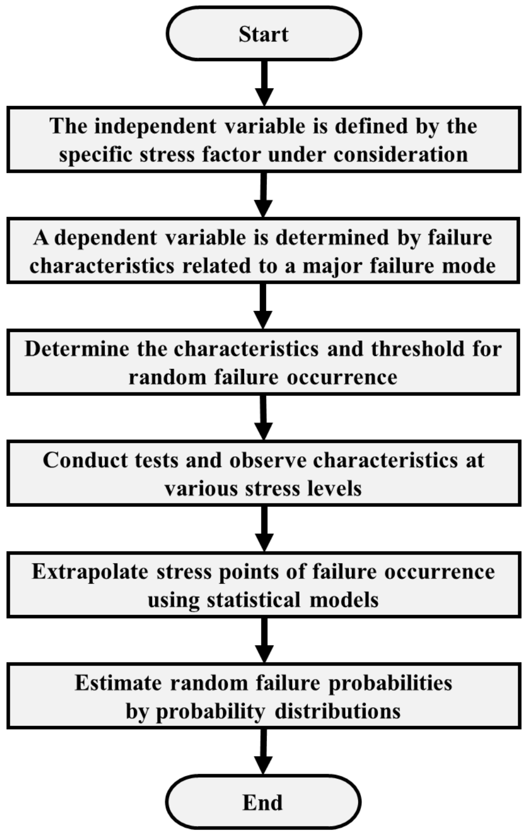

Figure 1. The Procedure of Test and Data Analysis

It is critical to recognize that stress predominantly influences the predominant failure modes. Specific tests are conducted to elucidate failure characteristics and their corresponding stress thresholds. In the graphical representation, stress levels are depicted on the horizontal axis, and failure characteristics on the vertical axis. A minimum of three stress measurements is mandated. Post-testing, both stress levels and failure characteristics are documented. Due to the extended test times leading to failures, failure stresses undergo extrapolation.

Experimental outcomes align with a regression model, estimating the failure stress at each threshold. Subsequently, these values conform to a probability distribution, whose goodness-of-fit is ascertained numerically through dual methodologies. The Pearson correlation coefficient, derived from least-squares estimation (LSE), is used to assess the linearity between the midline and the data points on the probability plot. Nonetheless, this coefficient remains incalculable when the gradient of the exponential distribution remains undivergent.

Incorporating MLE, the Anderson-Darling (A-D) test significantly emphasizes the distributional tails during the computation of the summed weighted squared distances between the fitted line and plotted points. The distribution with the minimal A-D statistic emerges as optimal. Generally, the MLE exhibits greater precision compared to the least-squares method. Subsequent plots of the probability distributions juxtapose the estimated stresses in line with the distributions. If data points align closely with the midline of a probability plot, the selected probability measure is deemed suitable. Finally, the failure probability at each stress level is determined by aligning the failure rate with the respective distribution.

3.2 Bayesian Inference

Bayesian inference, as illustrated in Figure 2, utilizes prior information to estimate parameters.

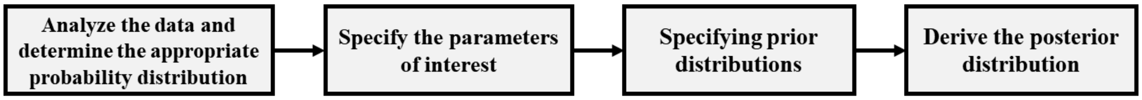

Figure 2. Estimation of Parameters When Employing Bayesian Inference

In this context, the posterior distribution is defined as follows:

$$P(\theta|x) = \frac{L(x|\theta)\pi(\theta)}{m(x)} \tag{1}$$

where $L(x|\theta)$, $\pi(\theta)$, and $m(x)$ are a likelihood function, a prior distribution, and a marginal distribution, respectively. Equation (1) shows the relationship between θ and the observed x values. The marginal distribution $m(x)$ describes the distribution of x, which may be discrete or continuous:

$$m(x) = \begin{cases} \sum_{\theta} L(x|\theta)\pi(\theta), \text{ if } x \text{ is discrete.} \\ \int L(x|\theta)\pi(\theta)d\theta, \text{ if } x \text{ is continuous.} \end{cases} \tag{2}$$

where $m(x)$ is a normalization constant independent of θ. Thus, the joint distribution $P(\theta|x)$ is:

$$P(\theta|x) \propto L(x|\theta)\pi(\theta) \tag{3}$$

The posterior distribution is calculated using Equation (1); at this stage, the prior distribution is determined by the likelihood function (the sampling distribution) (Hamada *et al*., 2008). The use of conjugate priors facilitates the calculation of the posterior distribution. Table 1 lists several conjugate priors commonly used when deriving likelihood functions. Probability density functions are employed to define the values of these parameters using prior data.

Table 1. Common Conjugate Priors (Hamada *et al*., 2008)

Likelihood Function	Conjugate Prior	Posterior Distribution
$Bernoulli(\theta)$	$\text{Beta}(\alpha, \beta)$	$\text{Beta}(\alpha + n\bar{x}, \beta + n(1 - \bar{x}))$
$Poisson(\lambda)$	$\Gamma(\alpha, \beta)$	$\Gamma(\alpha + n\bar{x}, \beta + n)$
$Geometric(\theta)$	$\Gamma(\alpha, \beta)$	$\Gamma(\alpha + n, \beta + n\bar{x})$
$Multinomial(\theta)$	$Dirichlet(\alpha)$	$Dirichlet(\alpha + n\bar{x})$
$Normal(\mu, \sigma^2)$ known μ	$Inv.\Gamma(\alpha, \beta)$	$Inv.\Gamma\left(\alpha + \frac{n}{2}, \beta + \frac{n(\bar{x} - \mu)^2}{2}\right)$
	$Inv.\chi^2(\nu_0, \sigma_0^2)$	$Inv.\chi^2\left(\nu_0 + n, \frac{\nu_0\sigma_0^2}{\nu_0 + n} + \frac{n(\bar{x} - \mu)^2}{2}\right)$
$Normal(\mu, \sigma^2)$ known σ^2	$Nomral(\mu_0, \sigma_0^2)$	$Normal\left(\frac{\frac{\mu_0}{\sigma_0^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}}, \frac{1}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}}\right)$

Conjugate priors facilitate the calculation of posterior distributions. However, their accuracy is not always guaranteed, as a posterior distribution might fail to adequately incorporate prior information. Table 2 provides a summary of the likelihood functions and conjugate priors commonly employed by reliability engineers.

Table 2. Conjugate Priors for Reliability Engineering (Hamada *et al*., 2008)

Likelihood Function	Posterior Distribution
$Exponential(\eta)$	$\eta \sim Inv.\Gamma(\alpha, \beta)$
$Weibull(\eta, \beta)$	$\eta \sim Inv.\Gamma(\alpha_\lambda, \theta_\lambda), \beta \sim \Gamma(\alpha_\beta, \theta_\beta)$ or $\eta \sim \Gamma(\alpha_\lambda, \theta_\lambda), \beta \sim Uniform(a, b)$
$Lognormal(\mu, \tau)$	$\mu \sim Normal(\mu, \sigma^2), \tau \sim \Gamma(\alpha, \beta)$

The observed values, denoted as $\mathbf{x} = (x_1, x_2, x_3, \cdots, x_n)$, include the i-th observed value, x_i. The probability density function features multiple parameters, i.e., $f(X = x_i)$ for $i = 1, 2, 3, \cdots, n$ where X is a random variable corresponding to the observed value x_i. The probability density function parameters are ψ and ω. The hyperparameters of each parameter are (ψ_1, ω_1) and (ψ_2, ω_2); their prior distributions π_1 and π_2 are those of Equation (4). The parameters of interest are ψ and ω, and the important hyperparameters are ψ_1, ω_1, ψ_2, and ω_2; all are greater than zero.

$$\psi \sim \pi_1(\psi_1, \omega_1), \omega \sim \pi_2(\psi_2, \omega_2) \tag{4}$$

The posterior distributions of ψ and ω are the product of the likelihood function and the prior distribution:

$$P(\psi, \omega; \boldsymbol{x}) = \frac{\pi_1(\psi_1, \omega_1)\pi_2(\psi_2, \omega_2)L(\psi, \omega; \boldsymbol{x})}{\int_0^\infty \int_0^\infty \pi_1(\psi_1, \omega_1)\pi_2(\psi_2, \omega_2)L(\psi, \omega; \boldsymbol{x})d\psi d\omega} \tag{5}$$

The marginal posterior distributions are derived from the joint posterior distribution $\pi(\cdot)$ of each parameter when estimating ψ and ω. The marginal distribution of ψ is:

$$P(\psi; \boldsymbol{x}) = \int_0^\infty P(\psi, \omega; \boldsymbol{x})d\omega = \frac{\int_0^\infty \pi_1(\psi_1, \omega_1)\pi_2(\psi_2, \omega_2)L(\psi, \omega; \boldsymbol{x})d\omega}{\int_0^\infty \int_0^\infty \pi_1(\psi_1, \omega_1)\pi_2(\psi_2, \omega_2)L(\psi, \omega; \boldsymbol{x})d\psi d\omega} \tag{6}$$

and the marginal distribution of ω is:

$$P(\omega; \boldsymbol{x}) = \int_0^\infty P(\psi, \omega; \boldsymbol{x})d\psi = \frac{\int_0^\infty \pi_1(\psi_1, \omega_1)\pi_2(\psi_2, \omega_2)L(\psi, \omega; \boldsymbol{x})d\psi}{\int_0^\infty \int_0^\infty \pi_1(\psi_1, \omega_1)\pi_2(\psi_2, \omega_2)L(\psi, \omega; \boldsymbol{x})d\psi d\omega} \tag{7}$$

These parameters can be estimated by computing the values expected from the marginal posterior distributions, represented by $\hat{\psi}$ and $\hat{\omega}$:

$$\hat{\psi} = \frac{\int_0^\infty \psi \int_0^\infty \pi_1(\psi_1, \omega_1)\pi_2(\psi_2, \omega_2)L(\psi, \omega; \boldsymbol{x})d\omega d\psi}{\int_0^\infty \int_0^\infty \pi_1(\psi_1, \omega_1)\pi_2(\psi_2, \omega_2)L(\psi, \omega; \boldsymbol{x})d\psi d\omega} \tag{8}$$

$$\hat{\omega} = \frac{\int_0^\infty \omega \int_0^\infty \pi_1(\psi_1, \omega_1)\pi_2(\psi_2, \omega_2)L(\psi, \omega; \boldsymbol{x})d\omega d\psi}{\int_0^\infty \int_0^\infty \pi_1(\psi_1, \omega_1)\pi_2(\psi_2, \omega_2)L(\psi, \omega; \boldsymbol{x})d\psi d\omega} \tag{9}$$

Equations (8) and (9) can be utilized to estimate ψ and ω, but double integration is often mathematically difficult. A numerical method, specifically Markov chain Monte Carlo (MCMC) sampling, is essential in this context. MCMC sampling uses Monte Carlo integration and a probability distribution to estimate novel probability distributions via random sampling employing a Markov chain; the mathematical characteristics remain unknown (Gilks and Richardson, 1995; Gamerman and Lopes, 2006; van Ravenzaaij *et al*., 2018). The Metropolis-Hastings (M-H) and Gibbs sampling algorithms are commonly used to generate Markov chains that estimate parameters via numerical analysis. The variance variable replaces point estimation; dimensional reduction via numerical integration is used to derive the marginal posterior distribution of the mean variable that affects the Bayesian inference. However, unbalanced sampling and variance heterogeneity may be apparent during comparisons with the estimated population means. To address this issue, Gibbs sampling does not generate approximations, instead suggesting methods of integration. A Gibbs approach was used to address high-dimensionality problems (Gelfand *et al*., 1990). Thus, an M-H algorithm primarily estimates single parameters; Gibbs sampling derives multiple parameters. During Gibbs sampling, a joint probability density function $f(x, y_1, y_2, y_3, \ldots, y_n)$ is used to yield the mean and variance of the marginal probability density function. However, often, the multiple integrations of Equation (10) are difficult or impossible. Gibbs sampling then simulates the joint distribution and yields the desired marginal distributions:

$$f(x) = \int\int\int \cdots \int f(x, y_1, y_2, y_3, \cdots, y_n) dy_1 dy_2 dy_3 \cdots dy_n \tag{10}$$

Gibbs sampling generates a sequence of samples $X_1, X_2, X_3, \cdots, X_n$ that adhere to the probability density function $f(x)$. If enough samples are created, the statistics are relatively precise (Casella and George, 1992). Given two random variables X and Y, Gibbs sampling generates the conditional distributions of $f(x)$ for X using the known conditional distribution of $f(y)$ for Y. This creates the following Gibbs sequence:

$$X'_0, Y'_0, X'_1, Y'_1, X'_2, Y'_2, \cdots, X'_k, Y'_k \tag{11}$$

To repeatedly iterate Equation (11), an initial value $Y'_0 = y'_0$ is set. Following that, alternate values for $Y_1, Y_2, \cdots, Y_n$ are created as shown below:

$$\begin{aligned} X'_j &\sim f\left(x \middle| Y'_j = y'_j\right) \\ Y'_{j+1} &\sim f\left(y \middle| X'_j = x'_j\right) \end{aligned} \tag{12}$$

For a sufficiently large j, a Gibbs sequence also yields approximate samples of $f(x)$ (Casella and George, 1992); the mean, variance, and quantiles of $f(x)$ can then be estimated.

3.3 Estimation of Random Failure Probability Based on Bayesian Inference

The design of tests for random failures and the estimation of their statistical probabilities have previously been discussed. However, there are instances where field reliability tests may be constrained by the availability of equipment and specimen numbers. Consequently, real-world estimations can deviate significantly from actual probabilities, particularly when data on stress levels are sparse. Extreme values can introduce bias into the estimated results. This section, therefore, applies Bayesian inference to test results obtained at specific stress levels, leveraging prior information. The assumptions are:

i. The analysis is confined to the target failure mode among various random failures.
ii. Failure characteristics are consistent, indicating genuine randomness in failure.
iii. The target item does not degrade over time.
iv. Absence of failure implies no degradation due to applied stress.
v. Probability distribution remains consistent across stress levels, implying that parameters shaping observed data distributions are statistically analogous.
vi. Accurate prior information exists for each stress level.

Below, the methodology for estimating the probability of random failure using Bayesian inference is outlined and illustrated in Figure 3.

In Section 3.1, frequency statistics are utilized in scenarios lacking prior information about the target item, as depicted in Figure 3. Conversely, this section outlines how the probability of random failure can be estimated using Bayesian inference when prior information is available. At this juncture, the data is numerical and comprises observed failure characteristics at different stress levels. Figure 4 illustrates this procedure.

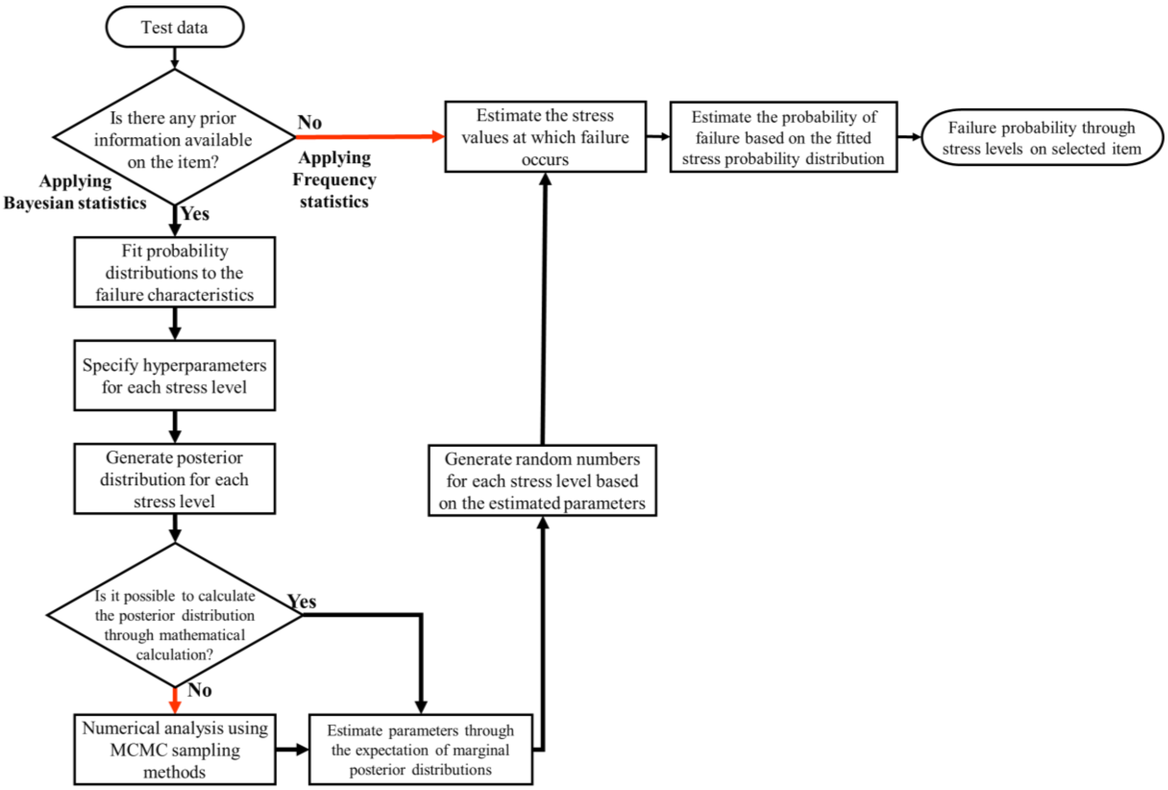

Figure 3. The Process of Estimation of Random Failure Probability Using Bayesian Inference

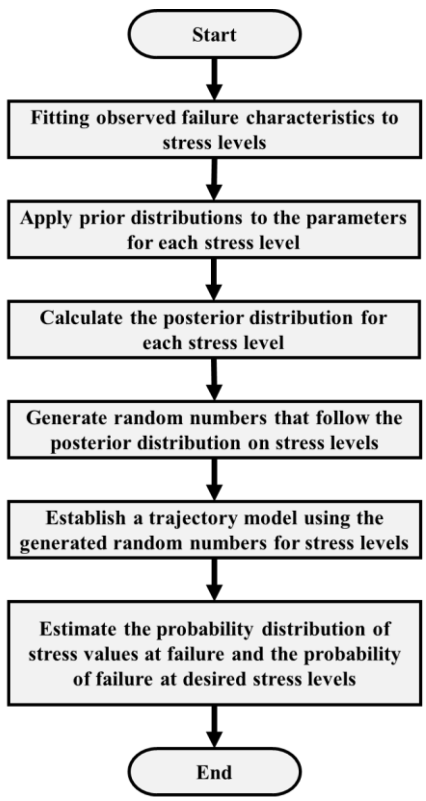

Figure 4. Procedure for Estimation of Random Failure Probability Using Bayesian Inference

The initial step in estimating random failure probability via Bayesian inference involves fitting the data to a probability distribution at each stress level. As outlined in Section 3.1, failure characteristics are determined for each stress level and then statistically fitted to a probability distribution. Should the failure modes remain consistent across all stress levels, the corresponding probability distributions will be identical. Consequently, the parameters of these probability distributions are statistically equivalent.

In the second step, prior information is applied to the failure characteristics at each stress level. The shape of the prior probability distribution is influenced by the chosen parameter. Engineers may select an appropriate prior distribution by referencing Tables 1 and 2 or relying on their experience. Hyperparameters for each stress level are established based on prior test results under constant stress. Before setting the hyperparameters for prior distribution estimation at each constant stress level, available prior information must be considered. The mean and variance hyperparameters are selected to reflect the precision of this information. Consequently, prior distributions for each stress level are derived.

The third step involves the calculation of the posterior distribution for each stress level using Bayes' theorem. The likelihood function, based on the observed failure data at that stress level, is combined with the prior distributions of the parameters. This combination is normalized to yield an updated parameter distribution, as depicted in Equation (3). This updated distribution then serves as the new prior for subsequent iterations of Bayesian inference. However, as indicated in Equation (5), most posterior distributions necessitate double integration and are not analytically solvable. To overcome this challenge, MCMC algorithms, such as Gibbs sampling, are employed, especially for multiparameter probability distributions. The mean of each parameter's posterior distribution can be estimated using Equations (8) and (9).

In the final step, random numbers based on the parameters obtained from the posterior distribution are generated. This process is visually represented in Figure 5.

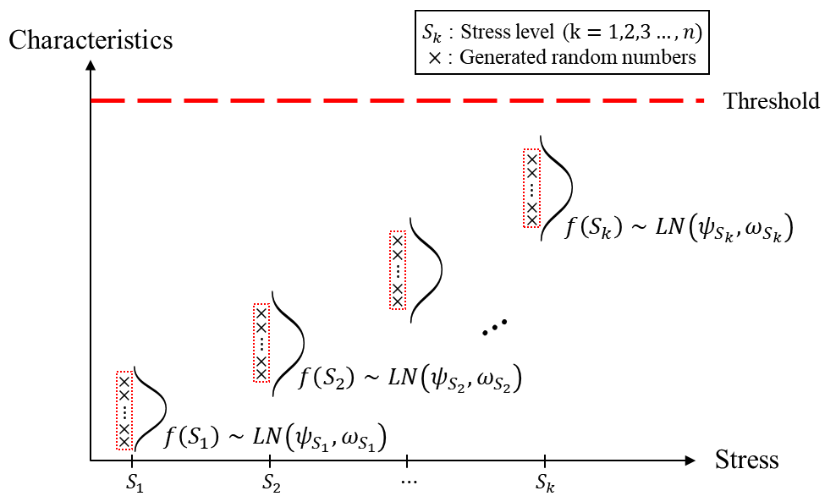

Figure 5. Random Number Generation for the Posterior Distribution at Each Stress Level

By generating a larger set of random numbers, the specimens increasingly approximate the probability density function of the estimated parameters, necessitating a sufficient quantity of these numbers. In the fifth step, random numbers generated for each stress level are organized in ascending order, and regression models are employed to estimate failure stresses at various threshold values. Optimization of each model's predictive power is achieved by minimizing the sum of squared errors (SSEs). Subsequently, failure characteristics at each stress level are extrapolated, allowing for the calculation of random failure probability using the derived probability distributions. Figure 6 depicts this fifth step.

The subsequent steps mirror those described in Section 3.1, wherein the probability of random failure is estimated using Equation (13) after fitting the estimated stress values to a suitable probability density function. For instance, $F(300) = 0.1$ indicates that at a stress level of 300 units, the probability of random failure is 10%, as determined by Equation (13). To clarify, S represents the random variable, while s_p denotes the stress level at which the probability of random failure is p.

$$F(s_p) = Pr\{S \leq s_p\} = p \tag{13}$$

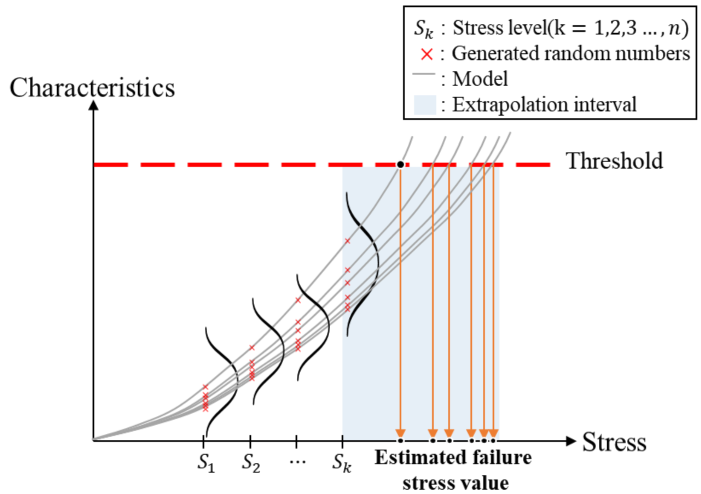

Figure 6. Regression Models Based on the Generated Random Numbers for the Various Stress Levels (Sim *et al*., 2022)

4. EXPERIMENTAL RESULTS

In this study, we estimated the random failure probabilities of brake discs using the proposed two-phase process. Given the critical role of brakes in automobile safety, understanding brake disc failures is paramount. These failures can be categorized by their mechanisms. Kao and Richmond (2000) identified one such mechanism as fracture at high rotor speed, leading to cold judder, while another type is high-temperature fracture, resulting in hot judder, which is more prevalent (Abdelhamid, 1997). Several methods have been developed to test for hot judder, including finite element analysis by Jung *et al*. (2011) and a simulation of the relationship between hot judder and heat transfer by Sim *et al*. (2013). Hot judder is classified as a random failure (Barber, 1969), whereas cold judder is a wear-out failure, testable through accelerated lifetime or degradation tests.

Using a Bayesian approach, we estimated the probability of random failure, drawing on results from random failure tests as discussed in Section 3.1. This innovative approach was compared with the MLE method, particularly in scenarios with a limited number of specimens. The failure stresses estimated for 1% of all product units from these tests were analyzed and contrasted. The results, obtained using our novel Bayesian process, underscored the effectiveness of Bayesian inference, contingent upon the accuracy and informativeness of the priors and the number of specimens used.

The Bayesian method, incorporating findings from the random failure tests in Sim *et al*. (2022), was utilized to estimate random failure probabilities. The estimated failure temperatures were aligned with a probability distribution using MLE, as shown in Figure 7. To fit the data to log-normal distributions, we employed both the probability plot and the Anderson-Darling (A-D) statistic.

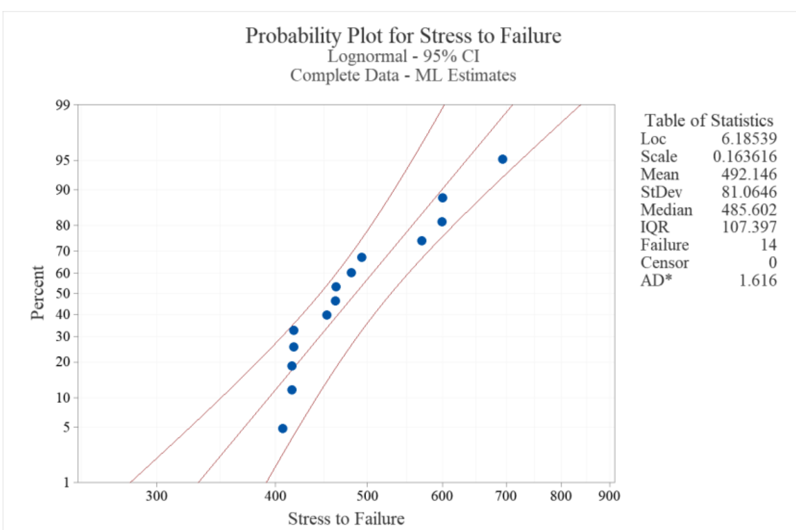

Figure 7. Failure Plots and Anderson-Darling Statistics for the Estimated Stress

For each stress level, the probability of random failure was determined using the log-normal distribution, with data derived from Figure 6 and Table 3.

Table 3. Random Failure Probability by Brake Disc Temperature

Temperature (℃)	300	350	400	450	500	550	600
Failure Probability (%)	0.16	2.27	11.80	32.08	57.09	77.67	90.20

4.1 Experimental Conditions

To facilitate a comparison between our innovative method and the MLE, we estimated the parameters of the log-normal distribution for each temperature stress level based on data from 18 specimens, as detailed in Table 4. The Anderson-Darling test was applied to each temperature stress level, and the results confirmed that the data conformed to log-normal distributions, consistent with the findings illustrated in Figure 7. Accordingly, Table 4 displays both the location and scale parameters of these distributions.

Table 4. Estimated Log-normal Parameters Used to Derive the Stress Level

Temperature (℃)	Location Parameter (μ)	Scale Parameter (τ)
150	0.3412	8.5898
200	0.4501	4.9361
300	0.3568	7.8551
400	0.3770	7.0359

Random numbers for the estimations were generated by fitting the log-normal distribution parameters. Noise, denoted as $\varepsilon \sim |N(0,1)|$, was added to each parameter to simulate real-world conditions. These adjusted parameters were then applied at every stress level, as outlined in Table 3. The generated random numbers served as the basis for subsequent estimations, with the details of the experimental approach presented in Table 5.

Table 5. The Experimental Conditions

<table>
<tr><th>Type</th><th>Number of Specimens</th><th>Prior Error</th><th>Hyperparameter $\left|\frac{\sigma^2}{\mu}\right| \times 100\%$</th><th>Gibbs Sampling (Iteration/ Burn-in)</th><th>Number of Random Numbers</th><th>Iterations</th></tr>
<tr><td>MLE</td><td rowspan="2">3,
5,
10,
15</td><td rowspan="2">+0%,
+50%.
+100%,
+150%,
+200%,
+300%</td><td>-</td><td>-</td><td rowspan="2">1,000</td><td rowspan="2">200</td></tr>
<tr><td>Bayesian Inference</td><td>+5%</td><td>20,000/
10,000</td></tr>
</table>

Considering the limitations of real-world testing, the number of specimens was set to 3, 5, 10, and 15 for different scenarios. Random numbers at each stress level were produced by fitting them to the log-normal distribution. Both the MLE and Bayesian inference methods were then employed multiple times to estimate probabilities under these conditions.

4.2 Effects of Hyperparameters on The Results

Based on data from 18 specimens, we estimated the temperature stress corresponding to a 1% random failure rate. The failure characteristics at this failure rate were extrapolated using an exponential model, which indicated a temperature of 331.88℃ as per the log-normal distribution parameters obtained through the MLE method. We compared the accuracy of estimations from both the MLE and our novel Bayesian inference method at this temperature stress level, considering specimen sizes of 3, 5, 10, and 15.

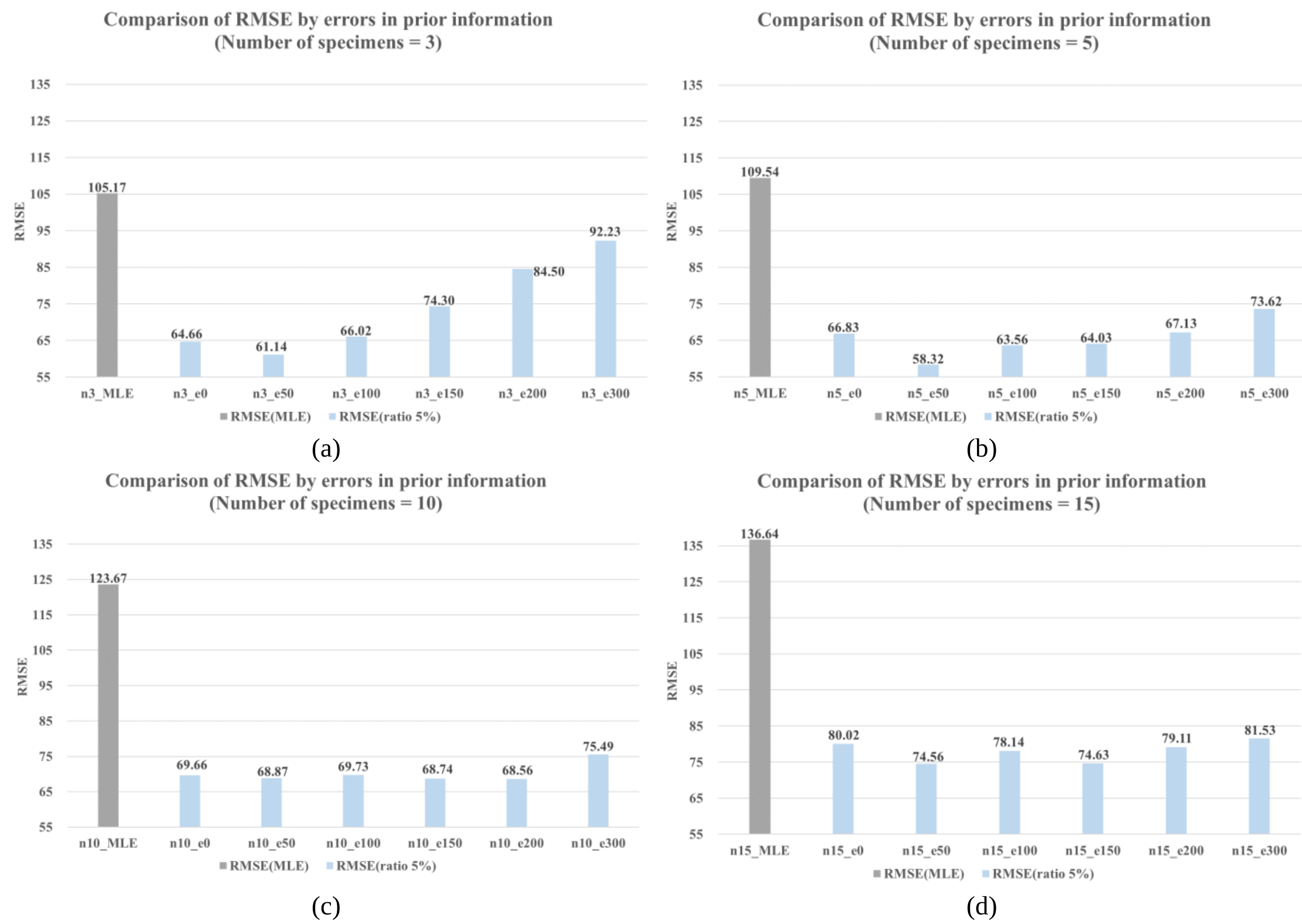

Figure 8. RMSE Values According to Prior Error

Employing data from 18 specimens, we estimated the temperature stress correlating to a 1% random failure rate. Failure characteristics extrapolated using an exponential model indicated a temperature of 331.88℃ for a 1% random failure, as determined by the log-normal distribution parameters via the MLE method. In this analysis, we evaluated the precision of estimations from both the MLE and our novel Bayesian inference method at this specific temperature. Specimen sizes of 3, 5, 10, and 15 were considered in this comparison.

Figure 8 illustrates the root mean square errors (RMSEs) in relation to prior information errors while maintaining a constant specimen count. The left side of the figure (in gray) represents MLE results across various data sizes, whereas the right side (in sky blue) shows Bayesian inference outcomes at a 5% hyperparameter ratio. The second graph from the left details the range of prior information errors, extending from 0 to 300%. Notably, as the error rate in prior information increased, particularly with a 5% hyperparameter ratio and lower specimen counts, the RMSE correspondingly rose. This increase in RMSE was more pronounced with fewer specimens, highlighting the significant impact of prior information errors. Such variability can be linked to the random sampling characteristic of Bayesian inference. Incorporating random numbers generated via the log-normal distribution, with added noise $\varepsilon \sim |N(0,1)|$ to each parameter, enhanced the estimation accuracy for both methods, especially as the specimen count decreased. Remarkably, Bayesian inference demonstrated superior accuracy over MLE, even with a 300% error in prior information.

When the RMSE was compared according to the hyperparameter ratio, it was observed that with a ratio of 5%, the RMSE tended to increase as the error of prior information increased. Therefore, it was concluded that when the prior information is not accurate, a benefit is derived from increasing the hyperparameter ratio to improve estimation accuracy.

In summary, the outcomes achieved using the proposed process demonstrated greater accuracy compared to those obtained via MLE. To enable a precise comparison of the experimental results, the ratios of RMSE for MLE and the proposed Bayesian inference-based process were evaluated, along with the RMSE values calculated at a 5% hyperparameter ratio of Bayesian inference. These comparisons are detailed in Table 6 and illustrated in Figure 9.

Table 6. Comparison of the RMSE Ratios Bayesian Inference to MLE

The number of specimens (prior error)	RMSE Ratios of Bayesian to MLE	The number of specimens (prior error)	RMSE Ratios of Bayesian to MLE
n3(e0%)	0.6148	n10(e0)	0.5633
n3(e50)	0.5814	n10(e50)	0.5569
n3(e100)	0.6278	n10(e100)	0.5638
n3(e150)	0.7065	n10(e150)	0.5559
n3(e200)	0.8034	n10(e200)	0.5544
n3(e300)	0.8770	n10(e300)	0.6104
n5(e0)	0.6101	n15(e0)	0.5856
n5(e50)	0.5325	n15(e50)	0.5457
n5(e100)	0.5803	n15(e150)	0.5719
n5(e150)	0.5845	n15(e150)	0.5462
n5(e200)	0.6128	n15(e200)	0.5790
n5(e300)	0.6721	n15(e300)	0.5967

Table 6 presents a numerical validation of the results previously discussed. The table's first column lists the number of specimens alongside prior errors. The RMSE ratios for Bayesian inference, set at a 5% hyperparameter ratio, are compared to those of MLE in the second and fourth columns. A ratio of 1 indicates RMSE values equal to MLE's at a 5% hyperparameter ratio. Ratios exceeding 1 suggest a higher RMSE with Bayesian inference, illustrating instances where it surpasses MLE in terms of RMSE.

Figure 9 illustrates the efficacy of the proposed method, as indicated by the data in Table 6. A clear trend emerges: the ratios decrease as the number of specimens increases, more so with fewer specimens and higher levels of prior error. This consistent trend of ratios below 1 underlines the superior performance of the proposed method, indicating that Bayesian inference is more effective for estimating random failure probabilities when specimen counts are lower.

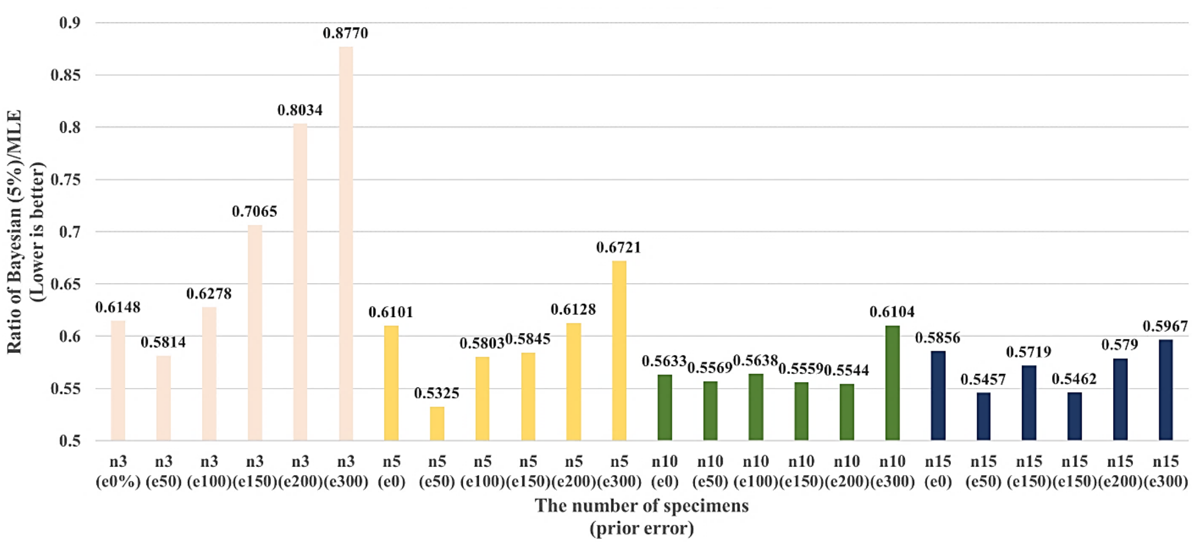

Figure 9. The RMSE Ratios Bayesian Inference to MLE

5. CONCLUSION

We introduced a method utilizing Bayesian inference, which effectively estimates the probability of random failures with limited data. Sim *et al*. (2022) presented a novel test design and statistical method for investigating such failures. Our findings suggest that random brake disc failures were primarily due to hot judder, as indicated by the tests conducted and the MLE methods employed. The reliability of test data is crucial, particularly when it is compromised by inadequate equipment or a limited number of specimens. Our Bayesian-based approach provides a cost-effective alternative for estimating the probability of random failures, particularly in scenarios where data collection is challenging.

In our evaluations, the Bayesian-based method surpassed the MLE in estimating the probability of unexpected brake disc failures, notably with datasets of equivalent size. Both graphical and statistical analyses were instrumental in highlighting factors influencing performance guiding the prioritization of tests under resource constraints.

The incorporation of prior test information significantly enhanced the accuracy of our random failure predictions, effectively offsetting the limitations posed by sparse data. Nonetheless, the quality of this prior information is critical, emphasizing the importance of reliable historical data.

Looking forward, we anticipate that the Bayesian-based process will facilitate more cost-effective estimation of random failure probabilities, especially in testing environments where data acquisition is a challenge. This method could be extended to statistical analyses of environmental tests that have not been conducted so far, assessing the failure probability of items under extreme conditions.

Future research directions include exploring the empirical Bayes approach, which depends on specific parameter values. In more complex scenarios, a hierarchical Bayesian inference could be implemented to determine hyperparameters, potentially addressing the underestimation of standard error caused by random hyperparameter effects. Our next step involves conducting statistical analyses on environmental tests and evaluating failure probabilities under extreme conditions. Additionally, while our research used identical specimens, further exploration into the reuse of non-destructed specimens is warranted, as items that undergo random failures may not degrade. This would enable optimal test design for random failures. Finally, although our study focused on brake discs using Bayesian inference, future work will extend to other areas, such as electronics.

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APPENDIX A – LIST OF NOTATIONS USED IN THIS PAPER

x : Observed data
θ : A universal parameter applicable to all probability distributions.
$P(\theta|x)$: A posterior distribution
$L(x|\theta)$: A likelihood function
$\pi(\theta)$: A prior distribution
$m(x)$: A marginal distribution
ψ : A specific parameter in a given posterior distribution
ω : A specific parameter in a given posterior distribution
π_1 : A specific prior distribution
π_2 : A specific prior distribution
ψ_1 : A specific parameter in π_1
ω_1 : A specific parameter in π_1
ψ_2 : A specific parameter in π_2
ω_2 : A specific parameter in π_2
$\boldsymbol{x}$: A vector includes $x_1, x_2, x_3, \cdots, x_n$
$P(\psi; \boldsymbol{x})$: The marginal posterior distributions of ψ
$P(\omega; \boldsymbol{x})$: The marginal posterior distributions of ω
$\hat{\psi}$: The expected value from the marginal posterior distribution of ψ
$\hat{\omega}$: The expected value from the marginal posterior distribution of ω
$F(\ )$: The probability distribution function characterizing the random failure of a specific item
s_p : A specific stress level
p : The failure probability at s_p